Fermionic quantum theory and superselection rules for operational probabilistic theories

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The anticommutation of Fermionic fields raises the problem of simulating the evolution of Fermionic systems by means of commuting quantum systems, say qubits. We tackle the issue considering local Fermionic modes as the elementary systems of an operational probabilistic theory. We show that the locality of Fermionic operations, namely operations on systems that are not causally connected must commute, implies the parity superselection rule, which inhibits the superposition of states with an even and an odd number of excitations. Accordingly we derive the largest probabilistic theory compatible with the parity superselection constraint and show that it lacks two distinctive traits of quantum theory, the local tomography and the monogamy of the entanglement. We generalize the notion of superselection rule to general probabilistic theories as sets of linear constraints on the convex set of states and prove a trade-off between the cardinality of the superselection rule and the degree of holism of the resulting theory. Within this scenario the Fermionic quantum theory and the real quantum theory can be regarded as superselected versions of the usual quantum theory. These results are published in Ref. [7] `http://iopscience.iop.org/0295-5075/107/2/20009` and Ref. [8] `http://www.worldscientific.com/doi/abs/10.1142/S0217751X14300257`.

In the last three decades the relation between Fermionic systems and other quantum systems has been throughly investigated from both the computational and the physical point of view. In particular the puzzling anti-commuting nature of the Fermionic systems casts a shadow on the possibility of simulating the physical evolution of a bunch of Fermionic systems by means of commuting quantum systems—say *qubits*. This issue was raised by R. P. Feynman in 1982 [9], when in his seminal work on physical computation he wondered about the possibility of simulating Fermions by local quantum systems in interaction—what we would call nowadays a *quantum computer*.

The problem is that of encoding the evolution of Fermionic fields onto localized quantum systems. A well-known encoding of N Fermionic systems into N qubits is given by the *Jordan-Wigner transform* (JWT) [14]. Such an encoding, based on the identification between the Fock space of N Fermions and the Hilbert space of N qubits, provides a $*$-algebra isomorphism between the Fermionic anticommuting algebra and the commuting algebra of qubits. Such a correspondence has been a valuable instrument in modern solid state physics for solving the one dimensional XY spin-chains [18, 15] and then for the understanding of superconductivity and the quantum Hall effect. Moreover, a *time-adaptive* JWT has been introduced in Ref. [19], which allows to contract Fermionic unitary circuits with the same complexity as for the corresponding spin model. In quantum information science the JWT has been used to extend to the Fermionic case notions as entanglement [1], the entropic area law [24], and universal computation [5]. More recently the JWT, which originally regards one dimensional chains of spin-$1/2$ systems, has been generalized to any spin [3] and lattice [13] dimension.

Despite its computational power, the JWT fails to solve completely the issue established by Feynman: physically local Fermionic operations are mapped into nonlocal quantum ones and viceversa. As noticed by many authors this can lead to ambiguities in defining the partial trace [16, 4, 17, 10], and in assessing

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the local nature of operations [22]. Independently on the JWT the Fermionic systems are usually assumed to obey the Wigner *parity superselection rule*. Based on the simple argument of the impossibility of discriminating a 2π rotation from the identity [20], this superselection rule corresponds to an inhibition to the superposition among states with an odd numbber and an even number of Fermionic excitations. Such a constraint on the admitted states for a set of Fermionic systems avoids the ambiguities connected to the JWT [1], but it has never been shown to promote the Jordan-Wigner isomorphism to a "physical isomorphism"—i.e.preserving some sort of locality of the Fermionic operations through the encoding.

In this work we tackle the issue of retaining locality of Fermionic operations through a qubit simulation in a novel way, namely considering the Fermionic modes as the elementary systems of an *operational probabilistic theory* (OPT). The context of OPTs provides a unified framework for studying and comparing properties of different probabilistic models, such as locality. Examples of OPTs are: (i) quantum theory (QT) (recently axiomatized within the operational framework [11, 6]), (ii) the classical information theory [6], (iii) the box-world [2], and (iv) the real quantum theory (RQT) [21, 12].

In Ref. [8] we build up the largest OPT corresponding to the Fermionic computation theory. We write all possible events (states, transformation, effects) of the theory achieved with the anticommuting algebra of the Fermionic field and assuming operations involving fields on some Fermionic modes to be local on those modes. Locality here is meant in the operational sense, namely operations on systems that are not causally connected must commute. The derivation leads to the parity superselection rule. Since there is not a unique OPT respecting such a superselection rule we then look for the largest theory compatible with the locality of Fermionic operations, here denoted Fermionic quantum theory (FQT).

Then we study the operational consequences of superselection. Unlike QT, FQT does not satisfy *local tomography*, i.e.the possibility of discriminating between two nonlocal states using only local measurements. After proving the correspondence between Fermionic and qubit local operations with classical communications (LOCC), we study the emerging notion of entanglement for Fermionic systems, an issue addressed in Ref. [1] for the first time. Here we identify non-separability as the unique notion of entanglement in FQT. Upon defining the Fermionic *entanglement of formation* and *concurrence*, we see that in FQT there are states with maximal entanglement of formation that are mixed and that Fermionic entanglement does not satisfy *monogamy of entanglement*, i.e.the limitation on the sharing of entanglement between many parties. Moreover the notion of maximally entangled state must be replaced with the one of *maximally entangled set* [23] also in the bipartite case, unlike QT. Interestingly, while in QT a simple linear criterion for full separability of states is lacking we will see that FQT allows for it.

A computational model based on Fermionic systems has already proposed by Bravyi and Kitaev in Ref. [5]. The model of Ref. [5] (that is proved to support *universal computation* and to be equivalent to the qubit computational one) coincides with the FQT with the additional constraint of *parity conservation*. As a consequence the resulting sets of transformations are strictly included in the FQT's ones. We compare QT and FQT from the point of view of computational complexity, and using the results in Ref. [5] we show the equivalence of the two theories and that even FQT supports universal computation.

It is worth noticing that FQT is only a special example of superselected QT while the notion of superselection of Ref. [7] allows for many other theories. In the letter [7] we have introduced a notion of superselection rule for a general probabilistic theory, corresponding to a linear constraint over the convex set of states. In this framework a theory that lacks local-tomography is called holistic [12] and we provide a link between the number of linearly-independent constraints and the degree of holism of the superselected theory. The present notion of superselection rule contains the FQT as a superselection of QT as a special case, but also includes other cases. Among them we can mention the case of RQT, which also lacks local tomography [12] and monogamy of entanglement [25], and the theory with *charge superselection*, which only admits superposition of states having the same particle occupation number.

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