Causal Structure as a Resource

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I, Matthew Wilson, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

Over the last decade processes with quantum causal structures have been studied in depth, they have been shown to give advantages in information processing, computational, and causal inference tasks. One process of particular interest has been the quantum switch, a black box which takes as an argument a bipartite non signalling process and outputs a process. In this report I first concretely explore the advantages of causally non separable processes and consider some more elaborate protocols involving superpositions of n-partide operators over arbitrary causal orderings. I then discuss a failed attempt to find a more intuitive way to reason about separability of outputs of switch operators. Then a discussion and definition for a resource theory of causal order, followed by some discussion of the most general setting in which one may speak sensibly about higher order processes. This report starts out treating resources practically, computationally, with the aim to gain intuition for how to approach the resources more formally.

Acknowledgements

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Chapter 1

Literature Review

Here we review tools results and frameworks relevant to the study of causal structure as a resource.

1.1 Indefinite Causal Structure

Given that the causal ordering of events in spacetime is dependent on the mass distribution in spacetime, and in quantum mechanics mass distributions may exist in superpositions, it seems conceivable that the causal orders of events in spacetime might exist in superpositions. To capture these possibilities and even less intuitive indefinite causal structures, the framework of process matrices has been developed [3].

In the Process matrix formalism it is assumed that quantum mechanics holds in local laboratories, no assumption is made about the causal relations between different laboratories.

To compute probabilities of outcomes inside laboratories, consistency is required with the algebraic structure of quantum mechanics. Consistency with the linear representation of probabilistic mixtures and coarse graining forces the process operator to be multilinear on quantum instruments. Using the Choi-Jamilkovski representation $M_j^{A_1A_2} \equiv [I \otimes M_j(|\phi\rangle\langle\phi|)]^T$ of the quantum operations $\{M_j\}$, one may write down the most general billinear map using a process matrix W.

$$P(M_i^{(A)}, M_j^{(B)}) = Tr[W^{A_1A_2B_1B_2} M_i^{A_1,A_2} M_j^{B_1,B_2}]$$

We can infer from the form of the process matrix, the causal structures with which our probability set is compatible. $W = I^{A_1}W^{A_2,B_1}I^{B_2}$ implies at most, correlations between the output of lab A and input of lab B (A quantum Channel) . $W = I^{A_2}W^{A_1B_1}I^{B_2}$ implies at most, correlations between inputs of labs A and B.The most general process matrix for a quantum theory with definite causal order and $A \leqslant B$ ought to be a combination of these two cases,

$$W^{A\leqslant B} = I^{B_2}W^{A_1A_2B_1} \tag{1.1}$$

We may imagine that we have a probabilistic mixture of a process compatible with $A \leqslant B$ and a process compatible with $B \leqslant A$. We call any such process matrix "Causally Seperable"

$$W_{CS} = pW^{A\leqslant B} + (1-p)W^{B\leqslant A} \tag{1.2}$$

Any bipartite process matrix that cannot be decomposed in this way is referred to a causally non-separable.

1.2 The Quantum Switch

An example of a causally non separable process is the quantum switch [4] which takes as an input two quantum channels N_1 and N_2, and outputs a new channel. According to a control qubit, the quantum switch in the state $|0\rangle$ applies channel N_1 before channel N_2 and in the control state $|1\rangle$ applies channel N_2 followed by channel N_1. The krauss operators for the output of the quantum switch are [5],

$$W_{ij} = K_i^{(2)}K_j^{(1)} \otimes |0\rangle\langle 0| + K_j^{(1)}K_i^{(2)} \otimes |1\rangle\langle 1| \tag{1.3}$$

The quantum switch of two unitaries N_1, N_2 is then,

$$S(N_1,N_2)(|\psi\rangle, |+\rangle) = \frac{1}{\sqrt{2}}(|0\rangle N_2N_1|\psi\rangle + |1\rangle N_1N_2|\psi\rangle) \tag{1.4}$$

1.2.1 Channel Discrimination

The quantum switch is a computational resource when compared to quantum circuits with open holes [6]. Given two unitaries known to either commute or anti-commute, the quantum switch can be used to distinguish between the two cases,

$$[N_1, N_2] = 0 \quad \text{or} \quad \{N_1, N_2\} = 0 \tag{1.5}$$

with only one use of each channel. The output of the quantum switch can be rewritten in the $\{|+\rangle, |-\rangle\}$ basis.

$$\begin{aligned} S(N_1,N_2)(|\psi\rangle, |+\rangle) &= \frac{1}{\sqrt{2}}(|0\rangle N_2 N_1 |\psi\rangle + |1\rangle N_1 N_2 |\psi\rangle) \\ &= \frac{1}{\sqrt{2}}(|+\rangle \{N_2, N_1\} |\psi\rangle + |-\rangle [N_2, N_1] |\psi\rangle) \end{aligned} \tag{1.6}$$

Measuring the output in the $\{|+\rangle, |-\rangle\}$ basis gives $|-\rangle$ with certainty if $\{N_2, N_1\} = 0$ and $|+\rangle$ with certainty if $[N_1, N_2] = 0$.

1.2.2 Activation of Classical Capacity

The quantum switch of two completely depolarizing channels N_1 and N_2 has a non-zero classical capacity [5].

$$\begin{aligned} S(N_1,N_2)(\rho \otimes |\phi\rangle\langle\phi|) = &\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes \frac{I}{d} \\ &+ \frac{1}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes \frac{\rho}{d^2} \end{aligned} \tag{1.7}$$

So when the channel is applied to ρ some information about ρ remains in the output. All this information is classical however, the quantum capacity of the channel is 0.

1.3 Category Theory

A category C consists of

- A collection of objects $Ob(C)$
- Morphisms (arrows) between those objects $f : A \to B$

- An associative composition operation $\circ$, such that for any pair $f_1 : A \to B$, $f_2 : B \to C$, of morphisms the composition is itself a morphism $f_2 \circ f_1 : A \to C$.
- An Identity morphism Id_A for each object A, such that $\forall B, C, f : A \to B$, $g : C \to A$, $f \circ Id_A = f$ and $Id_A \circ g = g$

Ignoring sizing issues we may refer to the set of objects $Ob(C)$, and between any two objects $A, B \in OB(C)$ the set of morphisms between A and B, denoted $hom(A,B)$.

A morphism $f : A \to B$ has a 2 sided inverse if $\exists$ g such that $f \circ g = Id_B$, $g \circ f = Id_A$

1.3.1 Functor

A functor $F : C \to D$ associates to each object $A \in Ob(C)$ an Object $B \in Ob(D)$, and associates to each morphism $f : A \to B$ a morphism $F(f) : F(A) \to F(B)$ such that,

- $F(f \circ g) = F(f) \circ F(g)$
- $F(Id_z) = Id_{F(z)}$

A functor captures the idea that a pattern of morphisms in one category can be represented or seen in another category. Although fullness and faithfulness [7] more capture this intuition, since between any two categories there is always a trivial functor sending all objects to an object in D, and all morphisms to I_D.

1.3.2 Natural Transformation

We may ask when two functors are really the same embedding of one category into another. A natural transformation $\alpha : F \to G$ between functors F and G, is a collection of morphisms $\alpha_Z : F(Z) \to G(Z)$ for each object $Z \in Ob(C)$, such that $G(f) \circ \alpha_A = \alpha_B \circ F(f)$. Two functors are naturally isomorphic, writen $F \cong G \iff \exists$ natural transformations $\alpha : F \to G$ and $\beta : G \to F$.

1.3.3 Symmetric Monoidal Category

One way to introduce the notion that a joint system is itself a system, is to require that there is a functor from the product category $C \times C$ [7] into itself

$$\otimes : C \times C \to C \tag{1.8}$$

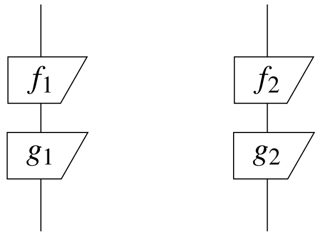


Figure 1.1: This picture represents both sides of equation 1.9

For a category to be symmetric monoidal (an SMC), there must exist a functor $\otimes$, there must be natural isomorphisms between $I\otimes A$, A, and $A\otimes I$, where I is a special object called the unit object, and there are some additional conditions on this functor and collection of morphisms called coherence conditions [8]. The conditions on an SMC imply

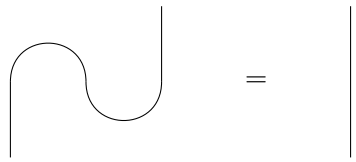
$$(f_1 \otimes f_2) \circ (g_1 \otimes g_2) = (f_1 \circ g_1) \otimes (f_2 \circ g_2) \tag{1.9}$$

Diagrams with morphisms as boxes and wires as systems can be used to reason in SMC's. Sequential composition is represented by the joining of wires between boxes, tensor composition is represented by placing the boxes next to each-other on the page. Equation 1.9 and and associativity are manifest in these pictures (figure **??**).

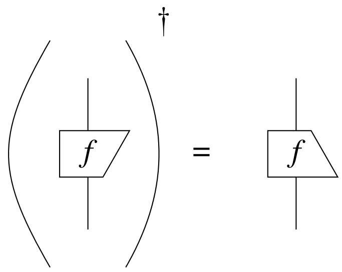
1.4 Categorical Quantum Mechanics

In categorical quantum mechanics $\otimes$ coincides with the standard tensor product of Hilbert spaces. Finite dimensional Hilbert space quantum mechanics is a dagger compact closed category [8], objects are Hilbert spaces, morphisms are linear maps. For each object A there is a dual object A^* which concides with the dual space in quantum mechanics. There exist special morphisms for each object

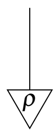
Which represent bell state and measurements, they satisfy,



which can be interpreted as post selected teleportation. When we write a morphism inside a trapezium, we can represent it's dagger by flipping the box in the horizontal plane.

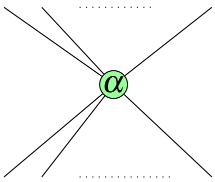


I typically models the behaviour of a singleton set in the category of Sets. Functions from singleton sets to other sets uniquely pick out elements of the codomain set. As such we may imagine that a morphism $\rho : I \to A$ represents an instance of the Hilbert space A, I.E a state. I is drawn as an invisible wire, and so states look reminiscent of rotated bra-ket notation.

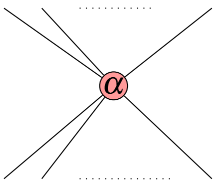


1.4.1 ZX Calculus

The ZX calculus is a graphical language for reasoning about qubit quantum mechanics [9], It consists of red dots blue dots and hadamard gates. The red dots correspond to sums in the computational basis, the green dots to sums in the fourier basis.



$$= \quad |00...0\rangle\langle 00...0| + e^{\alpha}\,|11...1\rangle\langle 11...1|$$



$$= \quad |++...+\rangle\langle ++...+| + e^{\alpha}\,|--...-\rangle\langle --...-|$$

They satisfy the axioms in 1.2, from which the antipode rule 1.4.1 can be derived

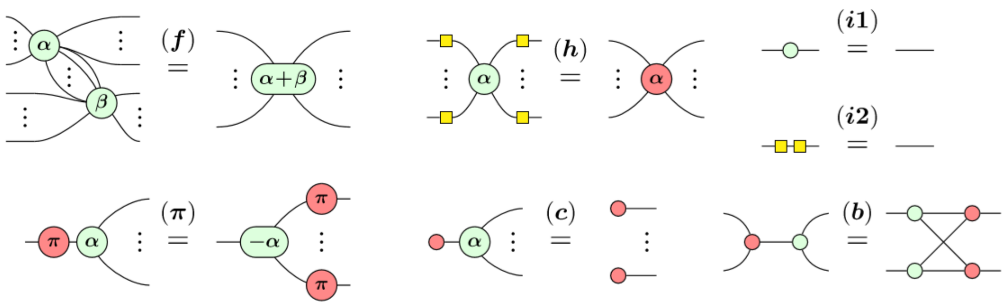
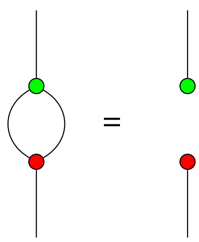
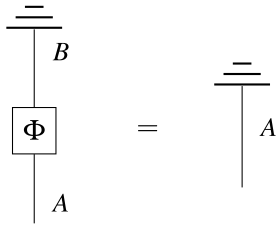


Figure 1.2: The rules of the ZX-calculus [1]



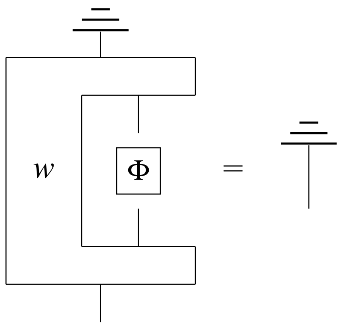
1.5 Causality in CQM

In quantum mechanics we take states with a particular normalization ($Tr[\rho] = 1$) to be the actual physical states that a particular system can take [10]. The only processes that can occur must be those which take as inputs, legitimate states, and output legitimate states, this is in quantum mechanics the trace preserving condition for a quantum channel. In the graphical language for CQM, the trace, (from here on referred to as the discard map), is given a particular symbol, and using it we may impose the above constraints (understood to be causality constraints) on morphisms [11]. Causal processes are those which satisfy,



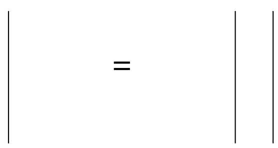
1.5.1 Second Order Causal SOC

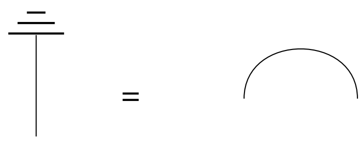
Second order causal processes send causal processes to causal processes, that is, a process is SOC if $\forall \Phi$ *causal*,



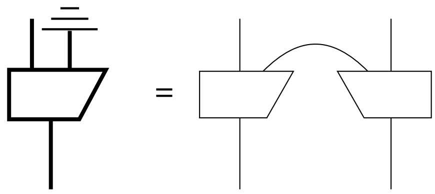
1.5.2 CPM

In this project frequent use of Selingers CPM construction is made. Each quantum degree of freedom, system, or object in CPM is represented by a pair of wires from HILB, one representing the bra of a density matrix, the other representing a ket.



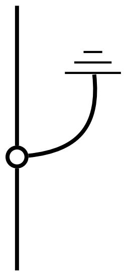


Any completely positive map can be considered as a unitary acting on an environment, followed by discarding that environment.



1.5.3 Classical Structure

We can represent the condition that a channel only lets classical information through by taking a quantum spider and applying the discard map to two of its legs, this is referred to as the decoherence map.



1.6 Categorical Semantics for Causal Structure

Without the linear algebraic interpretation of being the trace, one can sometimes still define a discarding map in a particular compact closed category. A category Caus(C) is defined in [12], which consists of

- Causal States
- Causal Processes (Proceses which take causal states to causal states)
- Causal Higher Order Processes (For example, operations which take causal processes to causal processes, such as a quantum switch)

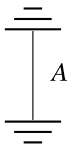
These processes may only be permitted to compose in ways which produce new causal states, processes, or higher order processes.

1.6.1 Precausal Category

The following definition is from [12]. A *precausal category* is a compact closed category C such that:

- C has discarding processes for every system, compatible with the monoidal structure

- For every (non-zero) system A, the *dimension* of A:

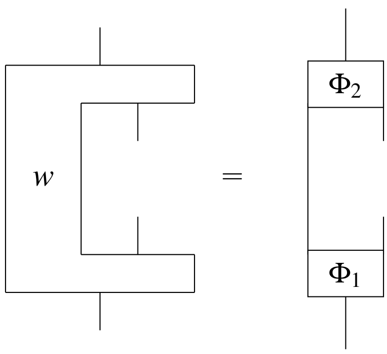


  is an invertible scalar.

- $\mathscr{C}$ has *enough causal states*:

$$(\forall \rho \text{ causal }, f \circ \rho = g \circ \rho) \implies f = g \tag{1.10}$$

- *Second-order causal* processes factorize: $\forall w \in SOC, \exists \Phi_1, \Phi_2$ causal such that



The circuit decomposition axiom turns out to be powerful in the context of this project. It also seems that Caus(C) can be defined without this assumption.

1.6.2 State Sets

For any set of states $c \subseteq C(I,A)$, the dual set $c^* \subseteq C(I,A^*)$ is defined as follows:

$$c^* \;:=\; \left\{ \pi : A^* \;\middle|\; \forall \rho \in c \;.\; \pi \circ \rho \;=\; 1 \right\}$$

$c \subseteq C(I,A)$ is closed if $c = c^{**}$

c is flat if there exist invertible scalars λ, μ such that

1.6.2.1 The Problem with Compact Closed Categories

Compact closed categories have process state duality, $hom(A,B) \cong hom(I, A^* \otimes B)$. Everything can be treated as a state. A state which encodes a process from (a process from A to B) to C is a state into the following object) [12].

$$\begin{aligned} (\text{Process})^* \otimes \text{State} &= (A^* \otimes B)^* \otimes C \\ &= B^* \otimes A \otimes C = \text{Process} \otimes \text{State} \end{aligned}$$

So the object, that a state is into is not enough to express what type of higher order transformation it represents.

1.6.3 Caus(C)

Caus(C) is a *-autonomous category constructed from C for which the objects states map into encode the type of causal higher order process they represent.

- Objects $\mathbf{A} \equiv (A, c_A)$. Where c_A is a closed flat set of states on A.
- Morphisms $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ are morphisms $f : A \to B$ such that $\forall \rho \in c_A, f \circ \rho \in c_B$

  the morphisms generalize the notion of being a causal map on their type.

1.6.3.1 Properties of Caus(C)

$c_{\mathbf{A} \otimes \mathbf{B}}$ is defined in [12]. $c_{A*} \equiv c_A^*$.

- Caus(C) is symmetric monoidal with respect to $\mathbf{A} \otimes \mathbf{B} \equiv (A \otimes B, c_{\mathbf{A} \otimes \mathbf{B}})$
- Caus(C) is star autonomous, with $(-)^* : C^{op} \to C$ defined on abjects by $\mathbf{A}^* = (A^*, c_{A^*})$
- $A \parr B \equiv (A^* \otimes B^*)^*$ is a second choice of tensor product.

1.6.4 1st Order Systems

A first order system is defined so that morphisms between first order systems in $Caus(C)$ are morphisms that are causal in C.

1.6.5 Lollipop Functor

One can confirm that for $\mathbf{A} \multimap \mathbf{B} \equiv (\mathbf{A} \otimes \mathbf{B}^*)^*$. The causal morphisms f : $\mathbf{I} \to (\mathbf{A} \multimap \mathbf{B})$ are isomorphic (via the cup and cap) to the causal morphisms in C. $\mathbf{A} \multimap \mathbf{B}$ represents the morphism type, and so Morphisms $\mathbf{w} : (\mathbf{A} \multimap \mathbf{A}') \to (\mathbf{B} \multimap \mathbf{B}')$ in Caus(C) are SOC morphisms in C.

1.6.5.1 Results concerning first order systems

For first order systems A_i,

- $\mathbf{A_1} \parr \mathbf{A_2} \cong \mathbf{A_1} \otimes \mathbf{A_2}$
- $(\mathbf{A_1} \multimap \mathbf{A_2}) \parr (\mathbf{B_1} \multimap \mathbf{B_2}) \cong (\mathbf{A_1} \parr \mathbf{B_1}) \multimap (\mathbf{A_2} \parr \mathbf{B_2})$.
- States $\rho : \mathbf{I} \to (\mathbf{A_1} \multimap \mathbf{A_2}) \otimes (\mathbf{B_1} \multimap \mathbf{B_2})$ are states representing non signalling processes, that is, the bipartite processes which are consistent with either causal ordering between them

Rather than speaking about $\mathbf{w} : \mathbf{I} \to \mathbf{X}$, we may speak about the underlying morphism w and declare its type $\mathbf{I} \multimap \mathbf{X}$.

1.6.5.2 SOC_2

A process is SOC_2 if it has type $((\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')) \multimap (\mathbf{X} \otimes \mathbf{C} \multimap \mathbf{C}')$, I.e it takes a non signalling bipartite process to a new process. SOC_2 processes are typically represented by I-shaped boxes as in figure 1.3. An example is the quantum switch, which for the $|0\rangle$ state pluged into system X routes maps in one order, and for $|1\rangle$, the other.

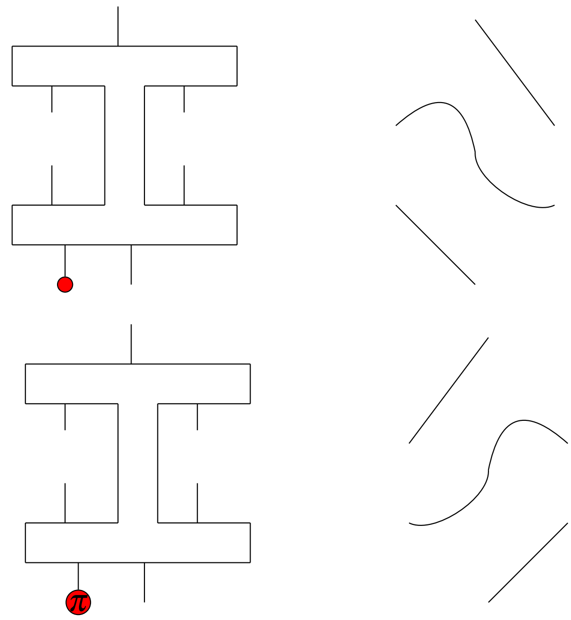


Figure 1.3: The quantum switch as a black box SOC_2 process

1.7 A Mathematical Theory of Resources

Any SMC can be interpreted as a resource theory [2] with

- Objects as resources
- Morphisms as free operations

SMC's can also be interpreted as process theories, with

- Systems as objects
- Morphisms as processes on systems

In [2] three recipes are given for turning a process theory C into a resource theory. First Define an all objects including subcategory $C_{free} \hookrightarrow C$ of processes considered to be free

1.7.0.1 Resource Theory of States

States of partitioned resource theory become the objects of $Ob(S(C,C_{free})) := \bigcup_{A\in|C|} hom(I,A)$. For morphisms $\eta \in S(s,t) \iff t = \eta \cdot s$.

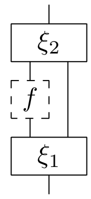


Figure 1.4: [2] A 1 comb is a pair of morphisms, partially connected by ancillary channel, with an open hole. The circuit decomposition assumption in a precausal category says that SOC decompose in 1-combs where both morphisms are themselves causal

1.7.0.2 Resource Theories of Parallel combinable processes

Morphisms in C become the objects in the resource theory.

$$|PC(C,C_{free})| = \bigcup_{A,B\in|C|} hom(A,B) \tag{1.11}$$

A transformation between two processes is performed by having the first process be part of a quantum circuit which is equal to the second process. The transformation itself is a circuit with a hole, and since free morphisms are closed under sequential and tensor composition, any circuit with a hole can be written in 1-comb form shown in figure 1.4

1.8 Quantum Superpositions of Causal Orders as an Operational Resource

In [13] a resource theory is defined for bipartite causally non separable processes, and some terminology is introduced. $B(H) \equiv \{$ Bounded Linear Operators on H$\}$.

- P = set of all legitimate process matrices $P \subset B(H_P \otimes H_{A_0} \otimes H_{B_0} \otimes H_F \otimes H_{A_I} \otimes H_{B_I} \otimes H_C)$
- CS = set of all probabillistic mixtures of causally ordered processes
- $W \setminus CS$ = set of causally non-seperable processes

The quantum switch is indefinite as a result of being the coherent control causal order, this is not the only sense in which a processes might have indefinite order.

1.8.1 Quantum Control of Causal Orders

Consider $W_{ABC} \in B(H_C \otimes H_{A_I} \otimes H_{A_O} \otimes H_{B_I} \otimes H_{B_0})$. Then W_{ABC} does not present coherent control of AB by C, if it can be expressed as a classical probabilistic distribution of control states and process matrices.

$$W_{ABC} = \sum_i q_i \rho_C^{(i)} \otimes W_{AB}^{(i)} \tag{1.12}$$

Let S be the set of all processes which have no coherent control of AB by C when any other systems are discarded.

$$NQC = conv(S \bigcup CS) \tag{1.13}$$

No quantum control of causal orders is the condition that either the control is in a product state with the actions on A and B, or there is simply no indefinite causal order. Two classes of transformations on process matrices are defined in 1001[13], both of which leave NQC invariant. A protocol is also given for distillation of generalized quantum switches.

Chapter 2

Practical Graphical Methods for the Quantum Switch

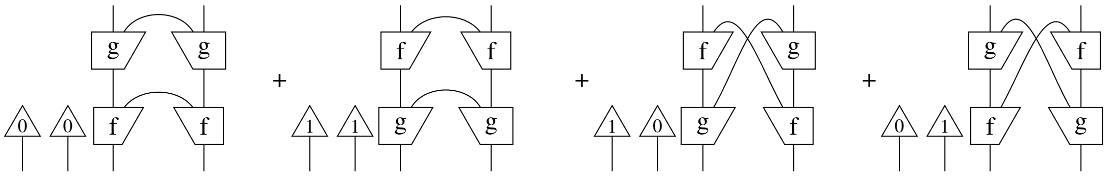
In this chapter graphical methods for generalized switches are explored, with an eye towards understanding the differences in capabilities of generalized switches for particular protocols.

2.1 Coherent Switch in the Computational Basis

The quantum switch can be split into four components

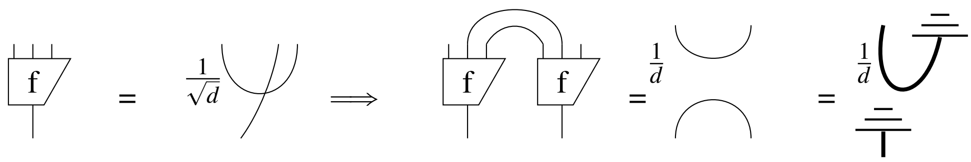
$$
\begin{aligned}
S(\rho,|+\rangle) = {} & \frac{1}{2}|0\rangle\langle 0| \otimes \sum_{a}^{d^2} \dots \sum_{f}^{d^2} K_a^{(0)} \dots K_f^{(0)} \rho K_f^{(0)^\dagger} \dots K_a^{(0)^\dagger} \\
& + \frac{1}{2}|1\rangle\langle 1| \otimes \sum_{a}^{d^2} \dots \sum_{f}^{d^2} K_a^{(1)} \dots K_f^{(1)} \rho K_f^{(1)^\dagger} \dots K_a^{(1)^\dagger} \\
& + \frac{1}{2}|1\rangle\langle 0| \otimes \sum_{a}^{d^2} \dots \sum_{f}^{d^2} K_a^{(1)} \dots K_f^{(0)} \rho K_f^{(1)^\dagger} \dots K_a^{(0)^\dagger} \\
& + \frac{1}{2}|0\rangle\langle 1| \otimes \sum_{a}^{d^2} \dots \sum_{f}^{d^2} K_a^{(0)} \dots K_f^{(1)} \rho K_f^{(0)^\dagger} \dots K_a^{(1)^\dagger}
\end{aligned}
$$

The same output can be written semi-graphically (semi since the expression still includes sums).



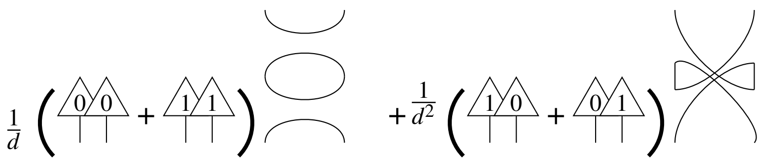
(2.1)

The bridges can be understood as implementing the sums over the Krauss Operators. The quantum switch of two completely depolarizing channels has a non zero classical capacity, this surprising result can be made intuitive in the graphical picture, where a completely depolarizing channel is written



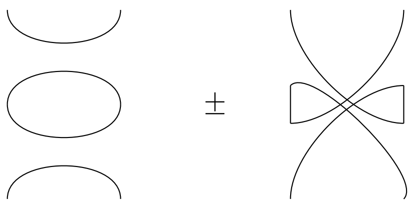
(2.2)

Which upon inserting into equation 2.1 gives



(2.3)

Measurement in the $|\pm\rangle$ basis will return a sum of two maps, up to normalization...



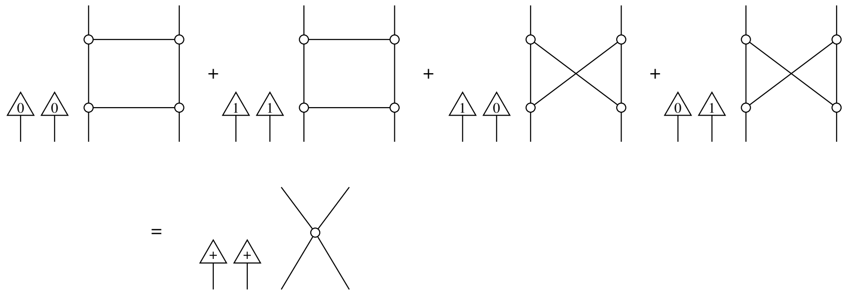
(2.4)

Measurement in the $|\pm\rangle$ basis will always return one of two superpoperators. By looking at these super-operators we can deduce for more general classes of maps, whether classical capacities are activated or de-activated.

2.1.1 Some Corollaries using Graphical Calculus

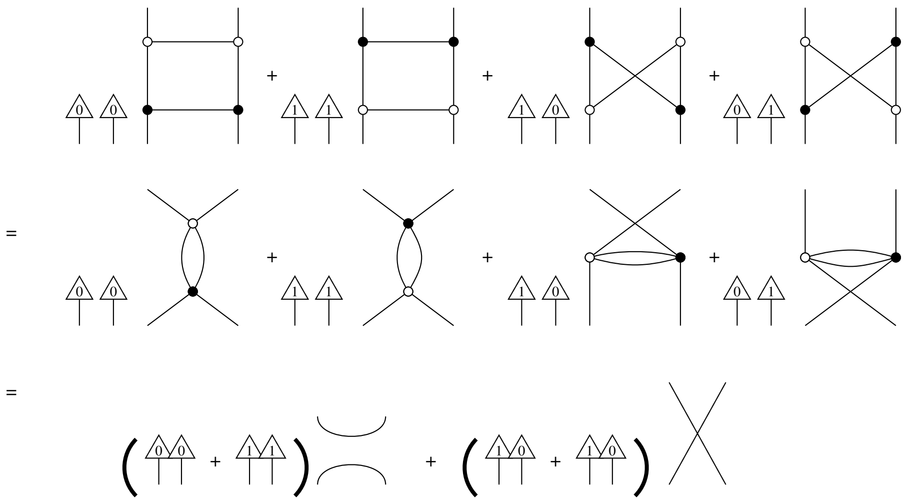
Corollary 1. The quantum switch of Decoherence maps in the same basis, is the same decoherence map.

Proof.



□

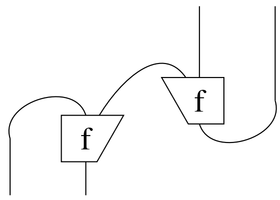
Corollary 2. The quantum switch of Decoherence maps in complementary bases,



By repeated use of the Bi-algebra equation, whilst definite order of the complementary decoherence channels is seperable, their coherent switch is not.

Corollary 3. The off diagonal elements of the quantum switch of an arbitrary map with the completely depolarizing map

$\cong$



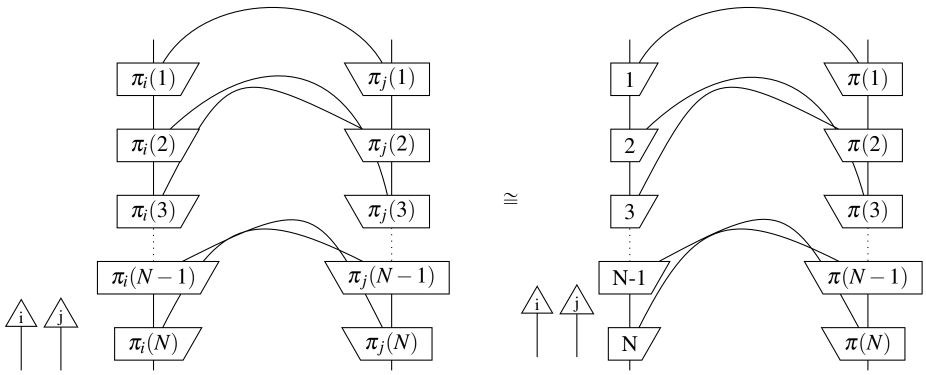
From which we immediately learn that the dimension of the space required for purification, (The trace subspace, the bridge), upper bounds the amount of information that can reach the bottom from the top.

2.2 $N_!$ Switches

Here we consider coherent control of multiple sequential orderings of multiple channels. Each additional degree of freedom in a control qudit could be assigned to a different causal order of input channels. For an N partite super-operator there are N! possible causal orders between the input holes. The graphical picture seems to simplify computations, and the search for useful protocols.

$$\rho' = S(\{i\})(\rho, \frac{1}{\sqrt{N!}} \sum_{i=0}^{N!-1} |i\rangle) = \frac{1}{N!} \sum_{ij} |i\rangle\langle j| \otimes \sum_{a}^{d^2} \dots \sum_{f}^{d^2} K_a^{(i_1)} \dots K_f^{(i_N)} \rho K_f^{(i_N)\dagger} \dots K_a^{(i_1)\dagger}$$

Formally each degree of freedom in the control corresponds to a particular permutation π of the open slots of the super-operator. The term $|i\rangle\langle j|$ of the $N_!$-Switch in the computational basis will come with a diagram where on the left hand side the boxes have been rearranged to permutation π_i, and on the right hand side the order of the boxes corresponds to permutation π_j. The only thing which matters is the relative permutation between them, we denote this π.



(2.5)

Each term in the output take the form $|i\rangle\langle j| \otimes \hat{O}$ .By using permutations $\{i_1 \dots i_{N-1} i_N\}$ and $\{j_1 \dots j_{N-1} j_N\}$, to write down two permutations in cycle form $(0 j_N j_{N-1} \dots j_1), (0 i_1 \dots i_{N-1} i_N)$. The cycle decomposition of their product $C_{\pi_{ij}} = (0 j_N j_{N-1} \dots j_1)(0 i_1 \dots i_{N-1} i_N)$. The phrases "i,j are cds" are used here to mean "either i is cds sortable to j or j is cds sortable to i" [14].

- $\hat{O} \propto$ Depolarising Channel $\Longleftrightarrow$ 0,N are not in the same cycle of C_π $\Longleftrightarrow$ i,j are cds
- $\hat{O} \propto$ (Information transmitting term) $\Longleftrightarrow$ 0,N are in the same cycle of C_π $\Longleftrightarrow$ i,j are not cds
- The normalization of a term is determined by the number of cycles $c(\pi_{ij})$ in the cycle decomposition of permutation $C(\pi_{ij})$
- More precisely a term is multiplied by a factor $\frac{1}{d^N} d^{(c(\pi)-2)}$ if i,j are cds and a factor $\frac{1}{d^N} d^{(c(\pi)-1)}$ if i,j are not cds

The normalisation condition comes from noticing that the normalisation is given by counting closed loops in diagrams. Each bubble in a diagram gives a factor of d. The number of cycles is related to the number of bubbles in the following way

- For a cds permutation π, number of Bubbles $= c(\pi) - 2$
- For a non cds ("ncds") permutation π, number of Bubbles $= c(\pi) - 1$

2.2.0.1 Proof

Define the following permutations in cycle notation.

$$X = (0i_1 \ldots i_{N-1} i_N)$$

$$Y_\pi = (0 j_N j_{N-1} \cdots j_1)$$

We may without loss of generality take the left permutation to be the identity and consider the right hand side to be some arbitrary permutation π. For the cycle decomposition of their product

$$C_{\pi_{ij}} = (0\pi_N\pi_{N-1}\ldots\pi_1)(01\ldots(N-1)N)$$

and so $C_\pi(a) = \pi(\pi^{-1}(a+1)-1)$. With slight modifications interference diagrams represent the operation of C_π. We extend the interference diagram to have two extra fake depolarizing channel terms, from $0--0$ and $(N+1)--(N+1)$. Then $C_\pi(a)$ is computed by starting at slot a on the right hand side and then following the simply connected path down once, across, up and back across.

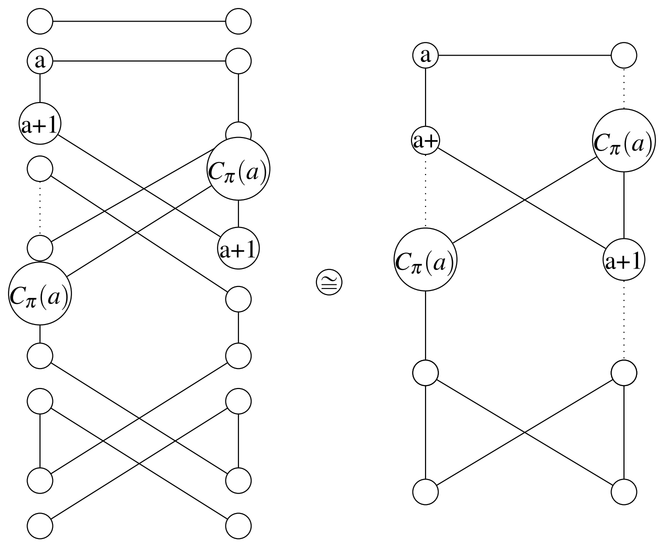
If The path taken starting at node 0, (and so entering the unmodified interference diagram from the top left) reaches node N (and so leaves the diagram through the bottom left) the term is proportional to the identity channel. It follows that if in the cycle decomposition of C_π, 0 and N are in the same cycle, the channel is proportional to the identity.

2.2.0.2 cds Permutations

cds permutations are posited to appear in nature. It can be proven that a permutation π is cds iff 0 and N are in the same cycle of C_π.

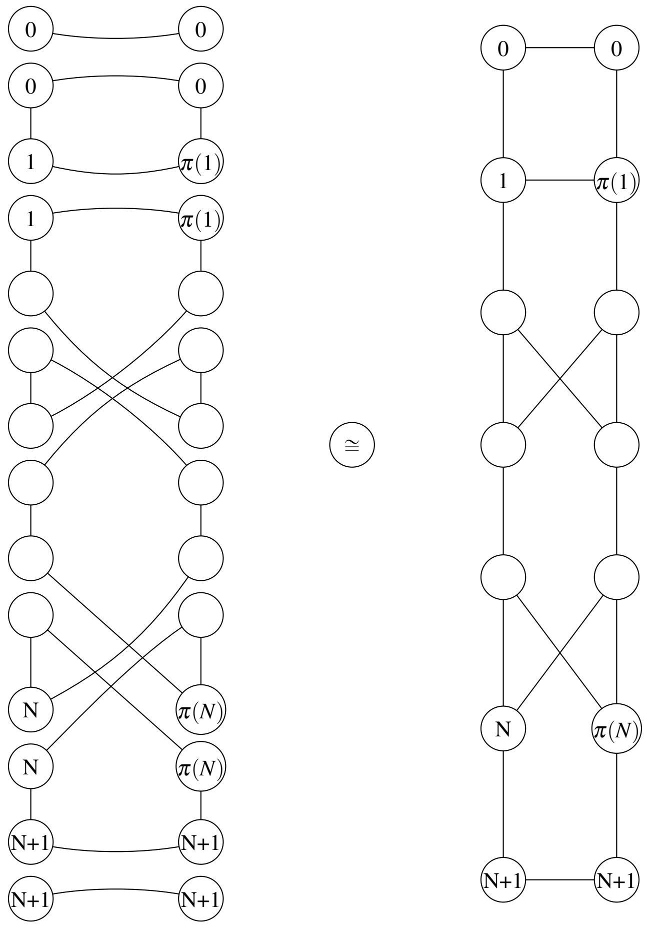
2.2.0.3 Remarks

- The cds permutations are not currently enumerated for general N, a suggested formula which fits untill N = 11 (The limit computed so far) is number of



(2.6)

Figure 2.1: Running a finger along an interference diagram is identical to computing $C_\pi(a)$



(2.7)

Figure 2.2: Modification to interference diagram, channels introduced at 0 and N+1

cds permutations (so number of depolarising terms) = $\frac{N!}{2} + \frac{(N-1)!}{2}$ for odd N. Note that this means the ratio between the number of information transmitting terms and the number of depolarising terms is less for odd N than it is for N=2

The general output of the quantum switch of N depolarising channels could be written semi-explicitly by defining,

$$cds_{i,j} \equiv \{\text{Pairs (i,j) of permutations which are related by cds permutations}\}$$

and

$$ncds_{i,j} \equiv \{\text{Pairs (i,j) of permutations which are not related by cds permutations}\}$$

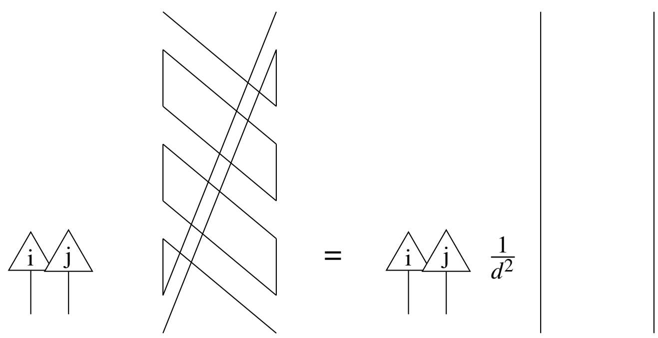
cds permutations give maximally mixed state, ncds permutations give information transmitting terms. So we can write

$$\rho' = \frac{1}{N!}\sum_{cds_{ij}} \frac{1}{d^N} d^{c(\pi_{ij})-2} |i\rangle\langle j| \otimes I + \frac{1}{N!}\sum_{ncds_{ij}} \frac{1}{d^N} d^{c(\pi_{ij})-1} |i\rangle\langle j| \otimes \rho$$

Where $c(\pi)$ is the number of cycles in C_π.

2.2.1 Cyclic N_C-Switch

Any term i, j for which, π_i and π_j are cyclic permutations (and hence cyclic permutations of eachother), give a term proportional to the identity.



(2.8)

Where the proportionality factor can be computed by counting loops in the diagram, the more loops the less suppressed the term is.

In this case we see that an interesting candidate for quantum N-Switch protocols is the superposition of the N cyclic permutations terms, since any two cyclic permutations of a list, are also cyclic permutations of each other. The general output is

$$\rho' = \sum_i |i\rangle\langle i| \otimes \frac{I}{d} Tr(\rho) + \sum_{i \neq j} |i\rangle |j\rangle \frac{\rho}{d^2} \tag{2.9}$$

2.2.2 Summary

The method has an intuitive graphical picture that allows one to guess good protocols, however sums of diagrams are awkward and make it difficult to deduce more generally under what conditions our super-operators produce channels with non zero classical or quantum capacities.

Chapter 3

How Much Time Travel can we Buy?

3.1 Quantum Switch and Post Selected Time-like Curves

Assuming a quantum $2_!$ switch can be implemented by a quantum circuit leads to the absrudity that the circuit can be composed with swap operations to produce a (post-selected) closed timelike curve [4]. With a quantum circuit and a post selected closed timelike curve, the quantum switch can be perfectly simulated. The quantum switch is equivalent to quantum circuits with time travel. This section generalizes these results to the $N_!$ Switch.

3.2 Switch of N! Causal Orders $\cong$ N-1 Post Selections

The proof is a generalization of the proof for the 2 channel case in [4]. Rather than using the superoperator formalism presented there, I use a description in terms of compact closed categories. The proof requires the use of ancillary channels of maps inserted into the quantum switch, so first I prove that it is legitimate to input part part of each of two 2-way signalling processes into a quantum N-Switch (lemma 5).

Lemma 4. $(\mathbf{A} \parr \mathbf{B}) \otimes (\mathbf{C} \parr \mathbf{D}) \hookrightarrow \mathbf{B} \parr (\mathbf{A} \otimes \mathbf{C}) \parr \mathbf{D}$.

Proof. Using the Canonical embedding $(\mathbf{X} \parr \mathbf{Y}) \otimes \mathbf{Z} \hookrightarrow \mathbf{X} \parr (\mathbf{Y} \otimes \mathbf{Z})$

$$\begin{aligned}
(\mathbf{A} \parr \mathbf{B}) \otimes (\mathbf{C} \parr \mathbf{D}) &\hookrightarrow \mathbf{B} \parr (\mathbf{A} \otimes (\mathbf{C} \parr \mathbf{D})) \\
&\hookrightarrow \mathbf{B} \parr ((\mathbf{A} \otimes \mathbf{C}) \parr \mathbf{D}) \\
&\cong \mathbf{B} \parr (\mathbf{A} \otimes \mathbf{C}) \parr \mathbf{D}
\end{aligned}$$

□

Lemma 5. For the Quantum N-Switch $\mathbf{w}$, the morphism in C underlying $\mathbf{w} \parr (\parr_{\mathbf{k}} \mathbf{I_k})$, appears in Caus(C) as $\mathbf{WI} : \bigotimes_{\mathbf{j}}^{\mathbf{N}} ((\mathbf{A_j} \multimap \mathbf{A_j'}) \parr (\mathbf{B_j} \multimap \mathbf{B_j'})) \to \mathbf{D} \multimap \mathbf{D}'$ Where all Capital letters in the preceding expression represent first order systems.

Proof. A quantum N switch $\mathbf{w}$ by definition appears as a morphism $\mathbf{w} : \bigotimes_{\mathbf{i}} (\mathbf{A_i} \multimap \mathbf{A_i'}) \to (\mathbf{C} \multimap \mathbf{C}')$. By functorality, $\mathbf{w} \parr (\parr_{\mathbf{k}} \mathbf{I_k}) : (\bigotimes_{\mathbf{i}} (\mathbf{A_i} \multimap \mathbf{A_i'})) \parr (\parr_{\mathbf{k}} (\mathbf{B_k} \multimap \mathbf{B_k'})) \to (\mathbf{C} \multimap \mathbf{C}') \parr (\parr_{\mathbf{k}} \mathbf{B_k} \multimap \mathbf{B_k'})$.

- First to show, is that there is a canonical map from the proposed input of $\mathbf{WI}$ to the input of $\mathbf{w} \parr (\parr_{\mathbf{k}} \mathbf{I_k})$. Let $\mathbf{X_i} \equiv (\mathbf{A_i} \multimap \mathbf{A_i'})$ and $\mathbf{Y_i} \equiv (\mathbf{B_i} \multimap \mathbf{B_i'})$. Assume an induction hypothesis for N

$$\bigotimes_{\mathbf{i}}^{\mathbf{N}} (\mathbf{X_i} \parr \mathbf{Y_i}) \hookrightarrow (\bigotimes_{\mathbf{j}}^{\mathbf{N}} (\mathbf{X_j})) \parr (\parr_{\mathbf{k}}^{\mathbf{N}} (\mathbf{Y_k}))$$

$\Longrightarrow$

$$\bigotimes_{\mathbf{i}}^{\mathbf{N+1}} (\mathbf{X_i} \parr \mathbf{Y_i}) \hookrightarrow ((\bigotimes_{\mathbf{j}}^{\mathbf{N}} (\mathbf{X_j})) \parr (\parr_{\mathbf{k}}^{\mathbf{N}} (\mathbf{Y_k}))) \otimes (\mathbf{X_{N+1}} \parr \mathbf{Y_{N+1}})$$

Then using Lemma 2

$$\begin{aligned}
&((\bigotimes_{\mathbf{j}}^{\mathbf{N}} (\mathbf{X_j})) \parr (\parr_{\mathbf{k}}^{\mathbf{N}} (\mathbf{Y_k}))) \otimes (\mathbf{X_{N+1}} \parr \mathbf{Y_{N+1}}) \\
\hookrightarrow &(((\bigotimes_{\mathbf{j}}^{\mathbf{N}} (\mathbf{X_j})) \otimes \mathbf{X_{N+1}}) \parr ((\parr_{\mathbf{k}}^{\mathbf{N}} (\mathbf{Y_k})) \parr \mathbf{Y_{N+1}})
\end{aligned}$$

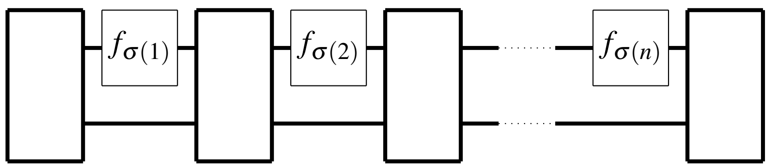
Lemma 2 also proves the base case.

- Finally, the output of $\mathbf{w} \parr (\parr_{\mathbf{k}} \mathbf{I_k})$ is already of type (First Order⊸First Order), since $(\mathbf{C} \multimap \mathbf{C}') \parr (\parr_{\mathbf{k}} \mathbf{B_k} \multimap \mathbf{B}'_{\mathbf{k}}) \cong (\mathbf{C} \otimes \bigotimes_{\mathbf{j}} \mathbf{B_k}) \multimap (\mathbf{C}' \otimes \bigotimes_{\mathbf{k}} \mathbf{B}'_{\mathbf{k}})$.

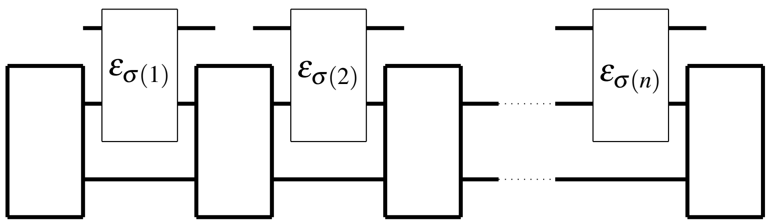
□

Theorem 6. *A circuit implementing the coherent superposition of the $N!$ orders between N quantum channels implies the possibility of travelling back in time $N-1$ times.*

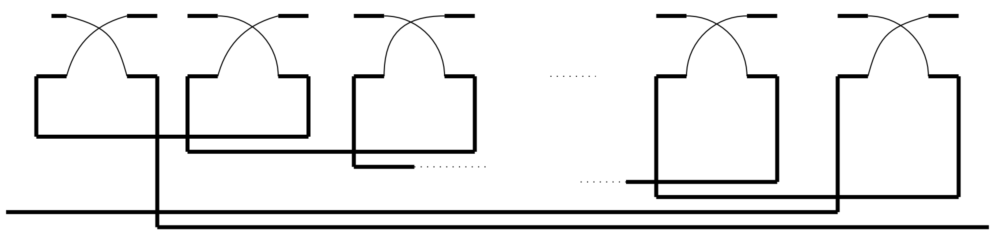
Proof. Identical to [4]. Any circuit containing 1 copy of each of $\{f_i\}$ can be written as



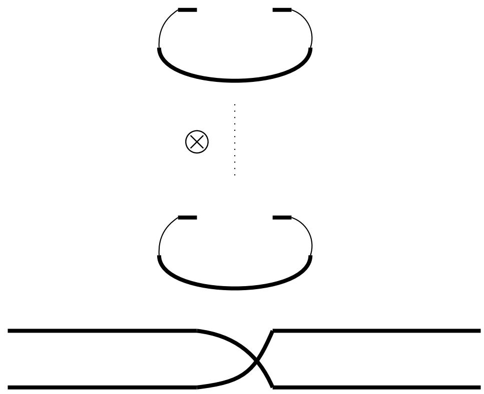
Without loss of generality we may relabel and take the permutation σ to be the identity. If the circuit implements one of the $N!$ permutations of the possible sequential compositions of the f_i for each of the $N!$ qudit states, then there exists some qudit state $|q\rangle$ for which the circuit must implement the map in figure().



The quantum switch as defined here acts in the same way independantly of whether the input maps have ancillary systems, and we have already proved that inserting ancillary maps is a legitimate causal operation. As in [4], choose each ε_i to be the swap



Then the quantum switch can be used to generate N time like loops.



$\square$

Theorem 7. *The Quantum $N_!$-Switch can be Implemented by a quantum circuit with ancillary channel $dim(H) = N!$, $2N^2$ Qudit controlled swaps, and N-1 Post selected timelike curves*

Proof. See string diagram in figure 3.1. Label each control degree of freedom by the permutation it should implement $|\pi_m\rangle$. Controlled swap $S^k_{i\alpha}$ with $i \in \{1 \dots N\}$, $k \in \{1 \dots N-1\}$, $\alpha \in \{0,1\}$, is implemented for all α by control qubit $|\pi_m\rangle$ if $i = \pi_m(k)$. For control qubit $|\pi_m\rangle$ this circuit implements $A_{\pi_m(1)}A_{\pi_m(2)} \dots A_{\pi_m(n)}$. $\square$

The utility in this picture is at least 2-fold.

- A concrete way to write down the ZX diagram for an Arbitrary $N_!$-Switch.
- It suggests that we count count the amount of indefinite causal order we have by the number of post selections needed to simulate the process. As such given a switch of (N-1)! orders, which is equivalent to N-2 post selected Bell states, we cannot hope to produce a switch of N! orders, which would require an additional post selection. Any formal resource theory of indefinite causal order should capture this fact. We however note that in principle a circuit with (N-2) post selected time-like curves might be able to produce a switch of $N! - 1$ or less causal orders. In fact, with one post selection and 4 qutrit controlled swaps, 5 of the 6 permutations of 3 channels can be implemented.

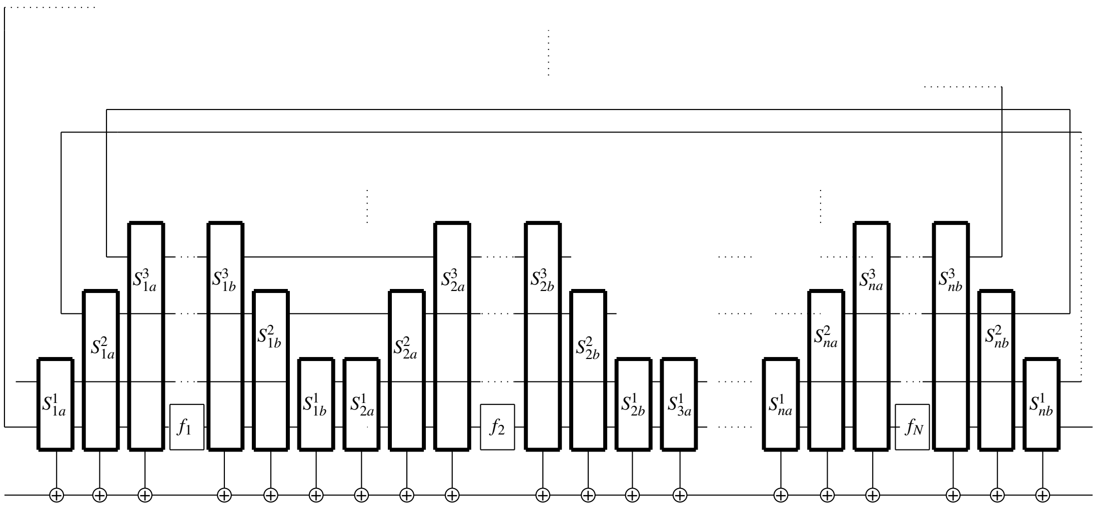


Figure 3.1: Quantum circuit with post selection which implements the coherent switch of $N_!$ orders between channels

It may be worth noting that for N nested quantum switches one can implement 2^N orders of N Channels.

3.3 N to N-1

Simply by inserting an identity channel into the N-Switch one may produce an N-1 switch, where the control qubit is now guaranteed to be in an $(N-1)!$ dimensional subspace. Such insertions of identity maps should also always come for free in any resource theory.

Chapter 4

Fully Connected Picture

4.1 Problems

So far graphical methods have been used to aid computation and intuition, however the properties of sums of diagrams are not always intuitive combinations of properties of the summands. A particular example is the semi-graphical computation verifying [15], which I have not presented here, the output of the quantum switch on an entanglement breaking channel can produce a channel with perfect quantum capacity, but it is very unclear that the sum of diagrams in the output of the quantum switch, even has any quantum capacity at all.

I wondered if we could deduce more generally some conditions for which our super-operators produce channels with non zero classical or quantum capacities.

4.2 Motivation - Controlled Unitaries

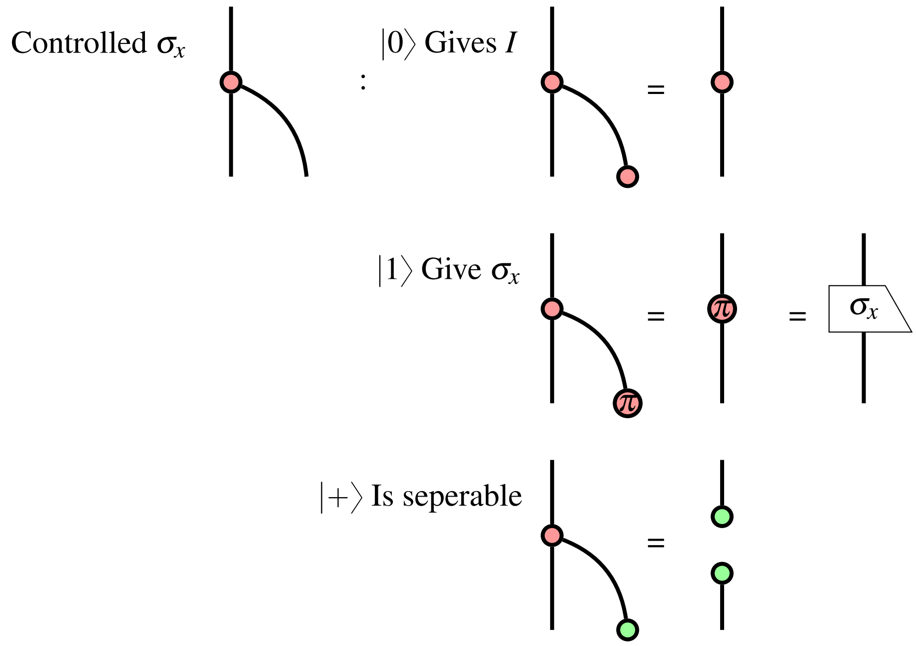
Given a black box controlled unitary in the computational basis, knowledge of each particular unitary does not necessarily give great intuition for the behaviour of the black box when given a control state that is in superposition of computational basis states (The same is not as true for probability distributions over computational basis states). To reason graphically with controlled unitaries in coherent control states, it seems that write the sum or look under the hood (write explicitly the contents) of the black box.

Similarly to reason graphically with (computational basis) controlled super-operators in coherent control states, without resorting to sums of diagrams, it seems we should look under the hood.

I found that the ZX calculus gave a way to reason about separability of controlled unitaries, even when supplied with a control in superposition. I hoped that this tool would transfer over to reasoning with coherent super-operators.

4.2.1 Toy Toy Unitary

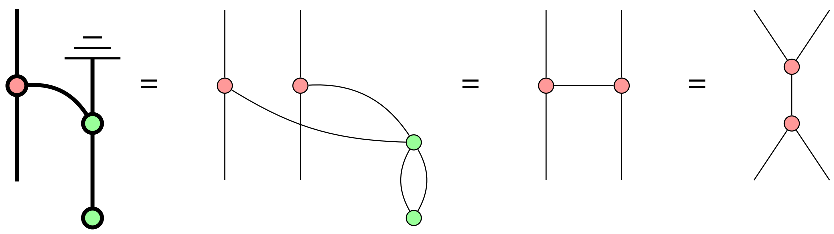
We can easily write down graphically a controlled σ_x or CNOT gate. For sure inserting the 0 state gives I and the 1 state gives σ_x.



(4.1)

Given this one fully connected graphical equation we may now reason about the effect of the plus state control, and we can immediately learn that the effect of the plus state is to turn the black box into a channel with 0 capacity, since it can be written as a separable map.

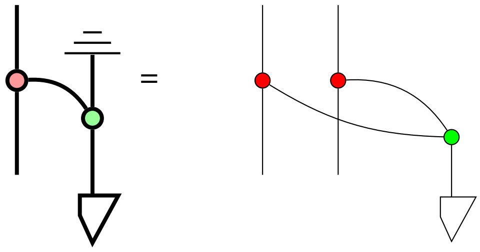
We can also reason about the equal probability distribution of the 0 and 1 states.



(4.2)

Here we see we get an entanglement breaking channel, one that has classical capacity but not quantum capacity.

This gives us some intuition for quantifying capacities graphically too, given our qubit controlled unitary, the less evenly weighted the probability distribution the higher the quanum capcity, and the probabillity distribution can be read off the diagram.

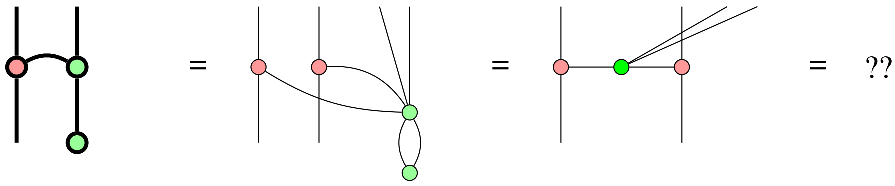


(4.3)

4.2.1.1 Open Ends

The standard definition of a CNOT has an open wire at the top which declares the value of the control qubit in the computational basis (Easily done since green copies red), we find that it is much harder to prove results about coherent controls when this open end is included.

Standard Representation of CNOT

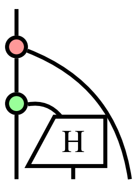


(4.4)

4.3 Toy Unitary

In a slightly less simple example the ZX calculus stil allows us to reason outside of basis control states. The Bi-Algebra rule becomes useful here.

We can write a coherent control of 4 Pauli channels as follows.



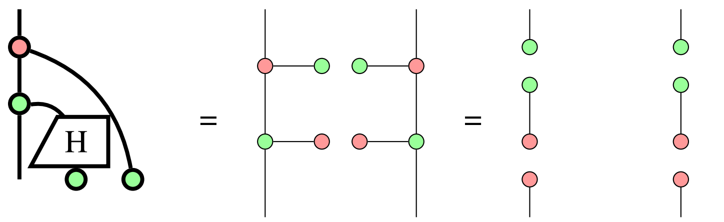
(4.5)

It is easy to check that for...

- Input state $|0\rangle|0\rangle$ the output is $\sigma_I\sigma_I$ (Perfect quantum and classical capacity)
- Input state $|0\rangle|1\rangle$ the output is $\sigma_I\sigma_y$ (Perfect quantum and classical capacity)
- Input state $|1\rangle|0\rangle$ the output is $\sigma_x\sigma_I$ (Perfect quantum and classical capacity)
- Input state $|0\rangle|0\rangle$ the output is $\sigma_x\sigma_y \propto \sigma_z$ (Perfect quantum and classical capacity)

Again we may now consider coherent control

- Input state $|+\rangle|+\rangle$ the output is separable (0 quantum and 0 classical capacity)



(4.6)

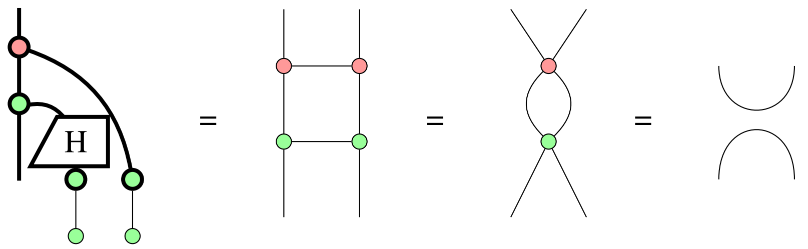
- Input state $|+\rangle|0\rangle$ the output is separable (0 quantum and 0 classical capacity), diagram omitted.

The ZX calculus also allows for reasoning with probabilistic control of Pauli channels,

$$C(\rho) = p_0\rho + p_x\sigma_x\rho\sigma_x + p_y\sigma_y\rho\sigma_y + p_z\sigma_z\rho\sigma_z \tag{4.7}$$

It is again easy to check that for,

- Both Input states having $p(o) = \frac{1}{2}$, $p(1) = \frac{1}{2}$ (Corresponding to $p_I = p_x = p_y = p_z = \frac{1}{4}$) the output is separable (0 quantum and 0 classical capacity)



(4.8)

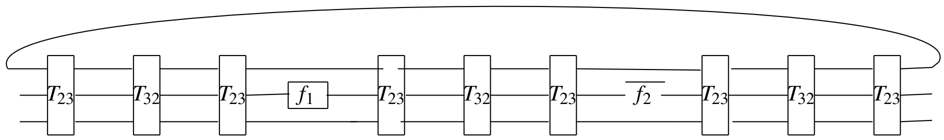
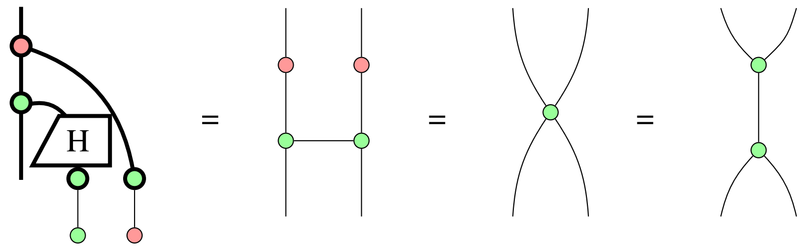


Figure 4.1: Post Selected quantum circuit implementing the quantum switch. T_{23} stands for a toffoli, when interpreted as a Controlled CNOT, is controlled by 1, and implements a CNOT with control on 2, target on 3

- Input states $p(0) = \frac{1}{2}$, $p(1) = \frac{1}{2}$ on the left and $p(0) = 1$ on the right (Corresponding to $p_I = p_y = \frac{1}{2}$), produce entanglement breaking channel

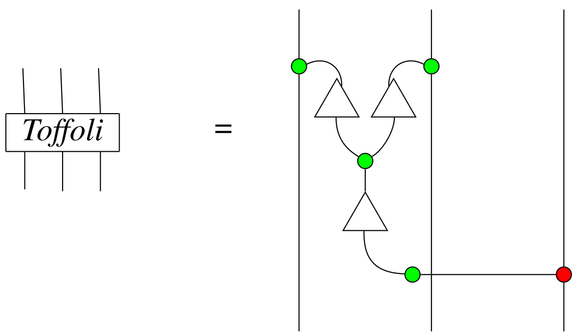


(4.9)

4.4 Fully Connected Quantum Switch in the ZX Calculus

By using the quantum circuit with post selection implementation of the quantum switch, one can look under the hood of the quantum switch, and write it down explicitly in a single ZX diagram (figure 4.1).

Using triangle nodes [16], a controlled CNOT (A Toffolli) can be written



(4.10)

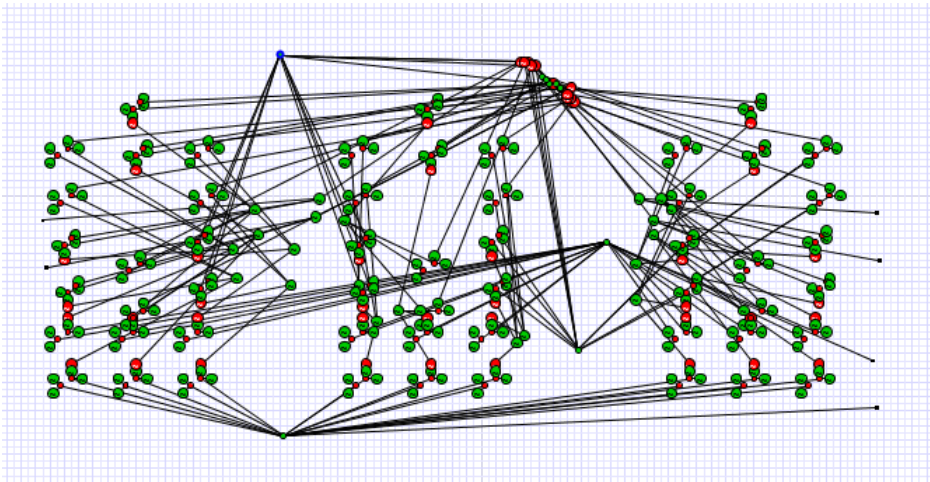
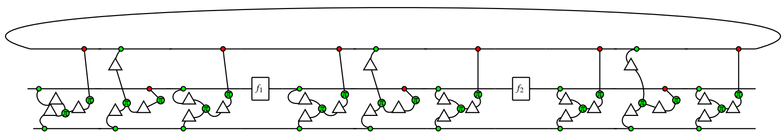


Figure 4.2: ”Simplification” of the quantum switch of two same-basis decoherence maps, little does quantum matic know, but this diagram is equal to a single node spider.

A controlled swap can be implemented with three controlled CNOTs.

So the following is a ZX diagram for the quantum switch.



We have one diagram, by doubling the diagram we can look at its result on any CPM, as one fully connected diagram. We can input the coherent control plus state, and see if the resulting ZX diagram is in any way illuminating.

4.4.1 Quantomatic

In a fit of laziness, I decied to ask quantomatic [17] to simplify the ZX diagram for me, the outcome was not illuminating. Chances are, this is because the automatic simplification procedures in Quantomatic explicitly deal only with circular nodes, so I had to expand out the triangle nodes explicitly in terms of red and green nodes. The next natural step then was to try to teach Quantomatic triangle rules, however these had to be given in terms of the ZX dot diagram for the triangle node too, since the current version of Quantomatic does not allow for theories with directional nodes.

Chapter 5

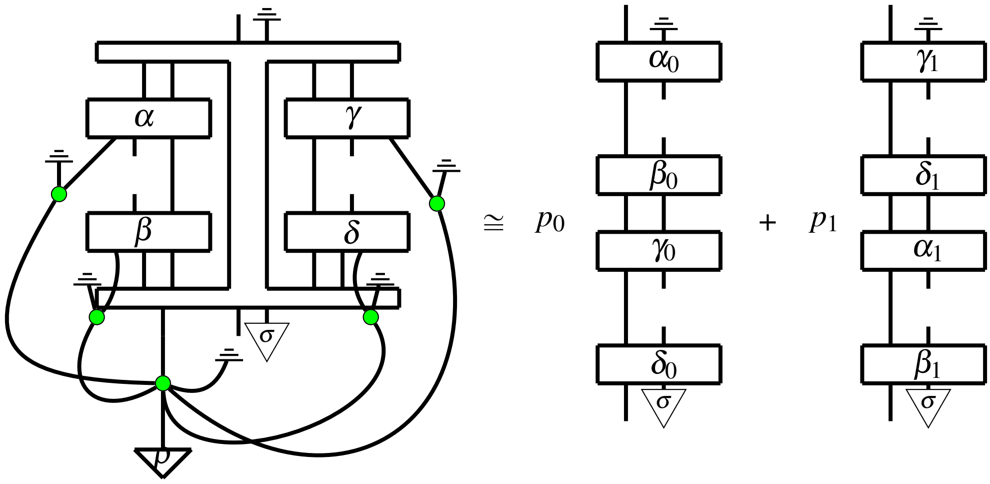
Causal Resource Theories

5.1 Motivation

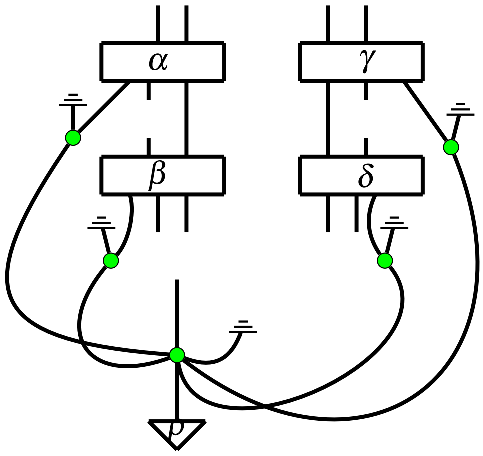
In the resource theory of entanglement [18], everything which does not generate entanglement is considered free, and all of those processes which are not composed of free processes are considered resources. With respect to the resource theory of entanglement, we are using the word process very generally, and including as processes, state preparation among things we consider to be processes.

5.2 Causal Non-Separability as a Resource Theory

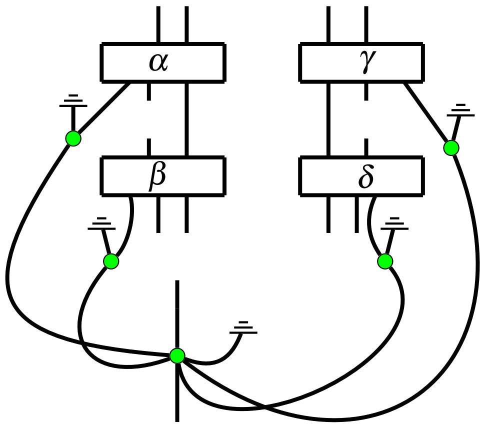
From a useful conversation with Aleks Kissinger we found a fully connected way to write the most general probabilistic combination of two oppositely linearly ordered bipartite combs.



In terms of resource theories, if the following class of operations are considered free

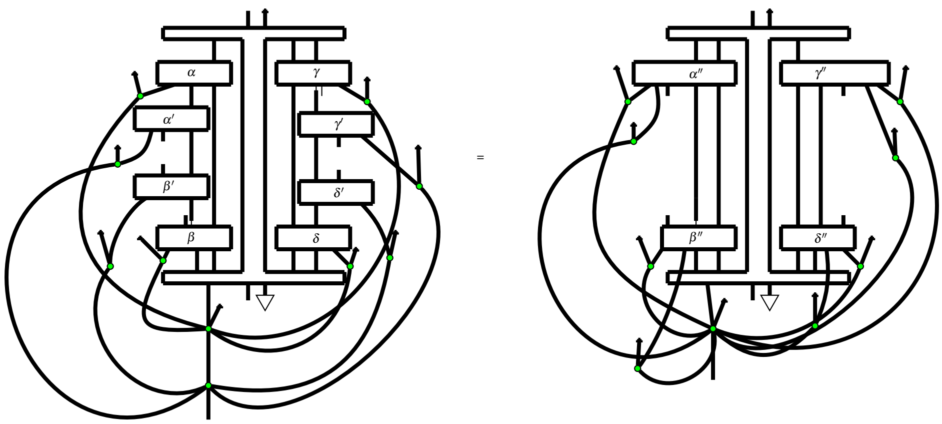


Then the quantum switch can be used to reach any such causally separable process. We should also expect



To be free.

We see that this process combined with our generally separable order switch, gives a new separably ordered switch



This picture is nearly useful, in future work it would be beneficial to look into purification's of process matrices, so that we can write all causally non seperable processes in this form.

5.3 Non Directed Order as a Resource

As a step towards a formal resource theory of indefinite causal order, we considered if we could use the types defined in a [12] to make a resource theory of non linear causal order. When the precausal category is CPM, we note that probabilistic mixtures of Linearly ordered super-operators would be included among-st the resourceful operations. I imagine that eventually one might be able to discuss at least four distinct flavours of causal structure in one framework.

- Linear Causal Orders
- Probabilistic Mixtures of Causal Orders
- Coherent Superpositions of Causal order
- (Causal) Inequality-Breaking Causal Orders

We have already shown we can write down quite general n-partide causally separable process as a single ZX diagram. It may then make sense to work directly with these pictures, that aside it seems interesting to ask what can be said by only referencing the type structure of Caus(C) directly.

5.4 Direct Definition of Resource Theory

We could try to go ahead immediately and define the SMC that is the resource theory, rather than starting with a raw materials SMC and using it to build a resource theory SMC. We take any Higher order process in $Caus(C)$ which takes a process compatible with a linear order to another process compatible with a linear order to be free. In words it seems to follow immediately that one cannot produce a non dag ordered process from these free processes. We now make this formal.

The definitions I make, typically take the morphism $w \in C$ to be the physical thing we may own, and so whenever we claim to have access to $\mathbf{w}$ we should also claim to have access to all of those morphisms $\mathbf{w}'$ such that $w' = w$.

5.4.1 Resource Theory of Perfect Linear Structure

By perfect I mean not even probabillistic.

$$C_{free} = Clos(\{\mathbf{f} : \exists \mathbf{f'} \in \mathbf{Caus}(\mathbf{C})(\mathbf{X},\mathbf{Z}) \text{ with } f' = f \text{ and } (\mathbf{X},\mathbf{Z}) = \bigcap_k Y_{(f)k}, \bigcap_k Y_{(f)k})\}$$

f and f' really have to be the same morphism in C, not just isomorphic morphisms.

Now we expect that any non direct resource theory of linear causal order we construct (See next section) should be a sub category of the close of the above.

5.5 Linear Order Type as a resource

Rather than jumping the gun and taking all of those Linear order preserving operations as resources, one could imagine a game where a participant may or may not be allowed some higher order operations, loosely call them nth order. In particular they are not allowed the resource operations, the Non Linearly ordered operations, and so for free they are allowed any operation compatible with a linear causal order.

Given 1 copy of a resourceful operation the player is tasked with constructing some other resourceful operation. With the tools currently available to them, they must use their free nth order processes to make an even higher order transformation that takes the nth order processes to nth order processes. A base case example is the resource theory of parallel combinable processes, where morphisms are used to construct circuits of morphisms with open holes.

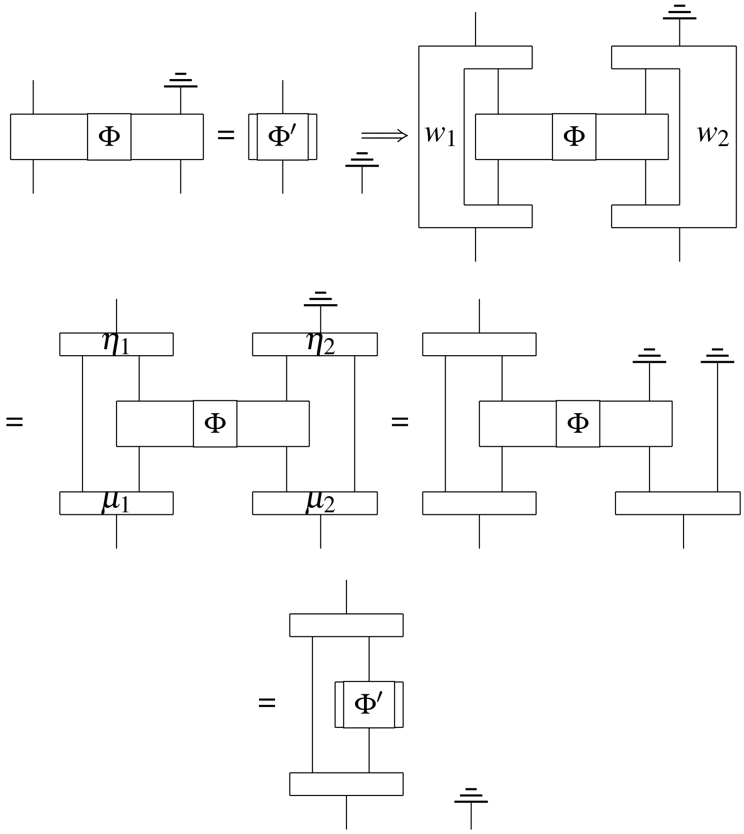
In the case of first order morphisms, there are only two ways to compose any two morphisms, in sequence and in parallel. We note that for a multi system morphism, plugging in some but not all wires really means to completely compose their $\otimes$ by identity extensions. By functorality it is immediate that for $\mathbf{f_1} : \mathbf{A_1} \rightarrow \mathbf{A_1'}$, $\mathbf{f_2} : \mathbf{A_2} \rightarrow \mathbf{A_2'}$,

- $\mathbf{f_1} \otimes \mathbf{f_2} : \mathbf{A_1} \otimes \mathbf{A_2} \rightarrow \mathbf{A_1'} \otimes \mathbf{A_2'}$
- $\mathbf{f_1} \parr \mathbf{f_2} : \mathbf{A_1} \parr \mathbf{A_2} \rightarrow \mathbf{A_1'} \parr \mathbf{A_2'}$

However, whilst it seems intuitive that when two tensor separable morphisms act on types, they should take linear orders between those types to linear orders between their output types, my current proof relies on a seemingly conceptually unrelated axiom, circuit decomposition.

Theorem 8. $\mathbf{w_1} : (\mathbf{A_1} \multimap \mathbf{A'_1}) \to (\mathbf{B_1} \multimap \mathbf{B'_1})$, $\mathbf{w_2} : (\mathbf{A_2} \multimap \mathbf{A'_2}) \to (\mathbf{B_2} \multimap \mathbf{B'_2}) \implies \exists \mathbf{w'_{12}}$ *such that* $w_{12} = w_1 \otimes w_2$ *and* $\mathbf{w_{12}} : (\mathbf{A_1} \multimap (\mathbf{A'_1} \multimap \mathbf{A_2}) \multimap \mathbf{A'_2}) \to (\mathbf{B_1} \multimap (\mathbf{B'_1} \multimap \mathbf{B_2}) \multimap \mathbf{B'_2})$.

Proof.



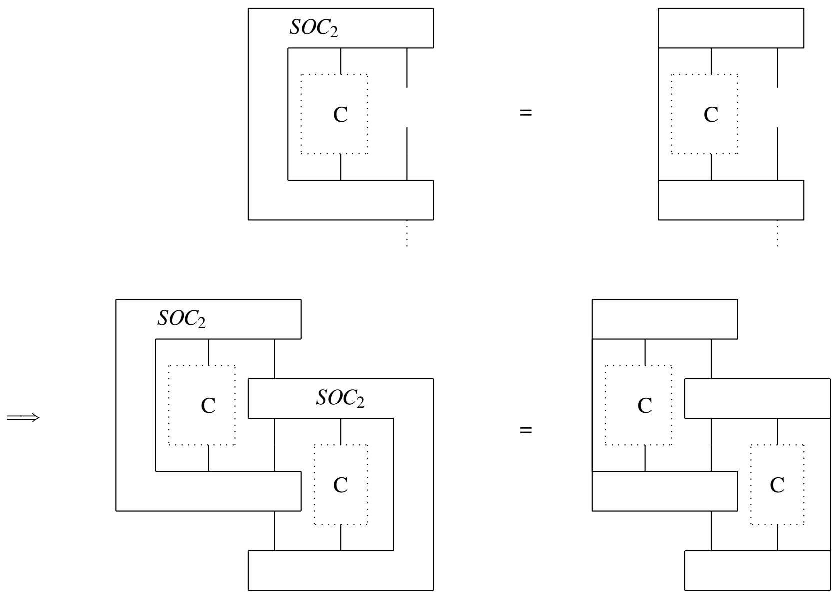
(5.1)

□

From the above theorem it follows immediately that tensor extensions with the identity take linear orders to linear orders. More alarming is the following

Theorem 9. *SOC_2 operations can be slotted inside each other to make legitimate SOC_2 operations.*

Proof.



□

Since it is hard to understand physically how to implement such a composition of higher order maps, we may find that in a resource theory of higher order maps, with circuit decomposition, that we make physical laboratory restrictions on methods of composition as well as placing restrictions on the processes themselves.

5.5.1 Definition of Resource Theory Of Linear Causal Structure

We could take any SOC_n morphism of the form $\mathbf{w} \in Caus(C)(\bigotimes_{\mathbf{i}}(\mathbf{A_i} \to \mathbf{A_i'}), \bigotimes_{\mathbf{k}}(\mathbf{C_k} \to \mathbf{C_k'}))$ such that $w : \mathbf{A_1} \to (\mathbf{A_1'} \to (\mathbf{A_2}...) \to \mathbf{A_n}) \to (\mathbf{A_n})' \to \mathbf{C_1} \to (\mathbf{C_1'} \to (\mathbf{C_2}...) \to \mathbf{C_n}) \to (\mathbf{C_n})'$. To be free. I would expect this class to be closed, but am unable to prove it. We can say that the free operations are the closure of this class, but it would be good to know explicitly what those morphisms are.

Chapter 6

General Higher Order Resource Theories

6.1 General Higher Order Resource Theories

For objects which we can imagine as representing Hilbert spaces we ask what is the most general way to include obects which represent maps between Hilbert spaces. If we imagine the morphism to represent a function we imagine that given a function and an element of the input we have all the data required to produce the output of the function [7]. That is there should exist some morphism called evaluation from $(X \multimap Y) \otimes X \to Y$. We see that to consider an object a morphism we should make sure that there category allows one to take products. We can define the function object by universal property, amongst all of those objects with evaluations maps.

6.1.1 Closed Symmetric Monoidal Category

Closure we enforces the existence of morphism objects, closure can be expressed as the requirement that the functors $- \multimap b$ and $- \otimes b$ are Left and Right adjoint functors of eachother.

6.1.2 Our Approach

One question that could be asked is whether morphism objects can exist under looser conditions, in an attempt to set the scene for the most general theory of higher order processes. We assume that we are in the setting of a symmetric monoidal category, we assume that we have some objects with labels such as $A \multimap B$, and we ask under what conditions we may consider these objects to really represent morphism types. Since states ought to represent instances of the object, they are an indirect way of speaking of the imagined elements of the object. We ask that states on $A \multimap B$ should correspond to morphisms in $Hom(A,B)$. As such there should be some notion of composition of two morphism states, which returns a new morphism state, corresponding to their morphism sequential composition. To make any progress we must assume a bijection $Hom(A,B) \cong Hom(I, A \to B)$. But we do not assume any naturality of the Bijection. Given any bijection we could in principle define sequential composition of morphism states.

$$\Delta_{abc} : [b,c] \otimes [a,b] \to [a,c] \tag{6.1}$$

We can view this as asking for there to exist a particular SOC2 morphism! Any such morphism compatible with a particular bijection can be written down.

$$\Delta_{123} \circ (F \otimes G) = \alpha_{13}^{-1}(\alpha_{23}(G) \circ \alpha_{12}(F)) \tag{6.2}$$

Our suspicion is that for Δ to be physically reasonable it should be the case that $\alpha_{(1)2}$ is a natural isomorphism between the Hom functors $Hom(I, A_1 \to A_2)$ and $Hom(A_1, A_2)$ on A_2 where we are considering A_1 to be fixed.

For the family of Δ morphisms to be compatible with braiding structure we should ask that

$$\Delta_{11'22'33'}((F \otimes F')(G \otimes G')) = \Delta_{123} \otimes \Delta_{1'2'3'}(I \otimes \sigma \otimes I)((F \otimes F')(G \otimes G')) \tag{6.3}$$

Which in turn places a condition on the Bijection

$$\alpha_{13}^{-1}(\alpha_{23}(G) \circ \alpha_{12}(F)) \otimes \alpha_{1'3'}^{-1}(\alpha_{2'3'}(G') \circ \alpha_{1'2'}(F')) \tag{6.4}$$

$$= \alpha_{11'33'}^{-1}(\alpha_{22'33'}(G \otimes G') \circ \alpha_{11'22'}(F \otimes F')) \tag{6.5}$$

Which in turn implies for $\alpha(F_I) = I$, $\alpha(F'_{I'}) = I'$.

$$\alpha(F_I \otimes F'_{I'}) = \alpha(F_I) \otimes \alpha(F'_{I'}) \tag{6.6}$$

Consistency with sequential composition is derivable from the bijection and so imposes no further conditions

$$\Delta(H,\Delta(G,F)) = \alpha_{34}(H)\alpha_{23}(G)\alpha_{12}(F) = \Delta(\Delta(H,G),F) \tag{6.7}$$

$$\implies \Delta(H,\Delta(G,F)) = \Delta(\Delta(H,G),F) \tag{6.8}$$

No more progress was made due to time constraints, in future work I hope to find the requirements on Δ which force α to be natural.

Chapter 7

General Conclusions

I have used mainly graphical methods to determine the properties of generalizations of the quantum switch. The purpose of exploring $N_!$ switches, was two fold, first to look for more specific protocols in which causal structure could be levied as a computational or information processing resource, and secondly to related higher and lower order switches, for the purpose of counting causal structure as a resource. Partial graphical methods with sums of diagrams were both computable and intuitive to a degree. Fully connected graphical approaches were less successful, although also less explored. We found via simple generalization of proofs in 1001[4] that to implement the superposition of $N!$ order requires N post selected time travels, as such, given N time travels, one could get more causal bang for their buck with an $N_!$ switch than they would by naively making N quantum switches. After some intuition for larger scale indefinite causal structures, some more formal work was attempted, first a general way of writing any probabillistic mixing of any two Combs ordered in opposite directions. That fact that these could be written entirely in terms of the switch, local unitaries, and decoherence channels, is a source of optimism that in many resource theories of indefinite causal order one could pin down important sections of resource theory pre-orders between resources with little work. A formal attempt to define a toy resource theory of linear order was made with reference to types in $Caus(C)$, the author found this to be frustratingly difficult and suspects that one should work directly with explicit expressions in C, although cannot articulate why other than to say that it has been difficult so far. Finally, having noticed that facts about preservation of causal structure in terms of type, were often easier to prove, or made possible to prove, using the circuit decomposition axiom, which the author again found supring. So in the most formal line of work the author considered working from the ground up, by defining an SMC, and trying to find the minimal condition under which one could regard

Chapter 8

Outlook and Future Directions

Just as with states and channels, some higher order processes are more resource-full than others, this has become apparent in the field of quantum Causal Structures. As such, the resources required to generate higher order resources, and the higher order resources needed to make lower order resources are questions of practical importance, independently of whether the more specific framework of indefinite causal structure can be implemented in near or far term laboratories. Future work and developments should involve a development of general higher order resource theories, in the spirit of [2]. In the specific context of indefinite causal structures, a hierarchy of non inequality breaking indefinite causal structures will be a contribution that allows researches to deduce what can and cant be achieved in specific situations. This may be particularly interesting in the context of resource conversion between higher and order lower resources, it has been recently shown that the quantum nature of gravity being tested in table top entanglement generation protocols [19] is the superposition of spacetime geometries [20]. Further explorations of resources generated by indefinite causal structures may provide us with further ways of extracting predictions from fundamental theories and further practical motivations to implement them, alongside related definite order protocols with similar properties, such as superpositions of trajectories [21]. The black box description of switches makes them more difficult to reason with, so further work should be done in understanding them with respect to explicit basis independent descriptions. Additionally it would be interesting to understand under what conditions inequality breaking indefinite causal structures [22] can be turned into non-inequality breaking ones.

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