Categorical Models for Quantum Computing

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After writing this dissertation, I have an even greater respect for everyone who works in this field. I find it amazing how they have transformed a branch of physics, that is conceptually very complicated, into such simple and elegant mathematics.

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Lastly, I want to thank my parents for their unconditional love, support and faith in me.

Abstract

Monoidal dagger compact categories were proposed by Samson Abramsky and Bob Coecke in [1] as a mathematical model for quantum theory. This is an alternative to, and a generalisations of, the Hilbert space model. Together with their colleagues, they have extended this to a versatile framework for quantum computing. Recently, Jamie Vicary has proposed an alternative framework in terms of weak 2-categories in [20]. In this dissertation, we will discuss these models. The focus is on the interaction of classical and quantum information, as this is particularly important for most quantum protocols.

In chapter 1 we will give a general introduction to categorical quantum mechanics. We discuss the basic concepts of quantum information and classical information; as well as different interpretations of quantum theory.

In chapter 2, we give an overview of the development of categorical models for quantum computing in terms of monoidal dagger compact categories. This includes the monoidal categories $CP(\mathbf{C}), Split^{\dagger}(CP(\mathbf{C}))$, $CP^{*}(\mathbf{C})$. Besides the technical details, we look at the interpretation of these models and how they represent different features of quantum theory, such as scalars, the Born rule, and classical information.

In chapter 3, we discuss weak 2-categories and their graphical language as models for quantum theory. These are a generalisation of the category $2Hilb$. We define a module theoretic construction $2(-)$ that turns a general monoidal category into a 2-category. This construction preserves most interesting properties of monoidal categories: dagger functors, compactness and biproducts. In addition, we propose a definition of dagger Frobenius algebras in 2-categories. This is a generalisation of dagger Frobenius algebras in monoidal categories. Furthermore, we describe a class of 2-categories, of which all 1-cells have canonical dagger Frobenius algebras. This is important for quantum computing because in these categories all states can be deleted. Finally, we work out in detail what the structure of the 2-categories $2(FHilb)$, and $2(Rel)$ is; and we construct an isomorphism between a full subcategory of $2(FHilb)$ and the weak 2-category $2Hilb$. The category $2(FHilb)$ is important, because $FHilb$ is the original model for quantum theory; and the category $2(Rel)$ is interesting, because Rel is a model for classical possibilistic computation.

In chapter 4, we give an overview and comparison of the categorical models we have discussed. We also point out some remaining open questions and interesting questions for further research.

Our contribution is:

- A module-theoretic framework, that provides a mathematical foundation for 2-categorical models of quantum theory. A detailed mathematical description of properties of this

model, including dagger functors, biproducts, compactness, dagger Frobenius algebras, and measurements.

- A detailed description of the category $2(FHilb)$, and an isomorphism between a full subcategory of $2(FHilb)$ and $2Hilb$.
- A detailed description of dagger Frobenius algebras and dagger modules on dagger Frobenius algebras in the category Rel; a description of the category $2(Rel)$.

Contents

Chapter 1

Introduction

In this dissertation, we will discuss several mathematical models for quantum theory. These models are important for the development of quantum computer science because they provide us with an abstract language to describe quantum processes, in which protocols are easily manipulated and checked for correctness. This gives us a high level of understanding of how quantum theory works and what we can do with it. Also, from the perspective of foundations of physics, it is interesting to explore models that generalize quantum theory, and relate it to other (physical) theories.

This work is part of the categorization program, which aims to find a categorical foundation for physics and mathematics. At the moment, this is still formulated in set theoretical terms. We assume familiarity with category theory, as well as with quantum mechanics. However, we will give a brief introduction to quantum information.

All lemmas are proven by the author, unless explicitly stated otherwise.

1.1 Mathematical models for quantum theory

The model that was originally proposed for quantum theory was formulated by Von Neumann, in terms of Hilbert spaces and completely positive operators on Hilbert spaces. This model has been successful in predicting outcomes of experiments, and its functionality is generally accepted amongst physicists. However, expressing and checking quantum protocols in this model involves matrix calculations, which can become rather complicated. By quantum protocols, we mean anything we can do using the rules of quantum theory, that is in some way interesting or beneficial. For instance: quantum algorithms, quantum teleportation or quantum key distribution.

Symmetric monoidal dagger compact closed categories were introduced by Abramsky and Coecke in [1], as an alternative model for quantum theory. This model generalises various aspects of Hilbert spaces. These categories naturally have a sound and complete diagrammatic language. This means that any well-typed equation in the model is provable from the axioms, if and only if its graphical representation is valid in the diagrammatic language. This makes it straightforward to reason about quantum protocols. As we will see, this model captures scalars, vectors, entanglement, classical structures, pure states, no-cloning, and the Born rule. It does not involve a generalization of the Schrödinger-equation, which describes how physical states change continuously with time, depending on their energy level. The model is based on the assumption that we can realise any energy level at any moment on the quantum system we are working with, and therefore it is possible to apply any (unitary) operation on this system. This is convenient in the sense that it makes the model simple and elegant. In the process of actually building a quantum computer, the Schrödinger-equation cannot be ignored. In practice, the establishment of some energy levels for our system can be challenging. Naturally, not everything that can be described by the model can be built (yet). However, for programming languages for quantum computing, we are only interested in whether or not there exists a transition from one state to the other, rather than whether this transition can be practically established.

Subsequently, variations on this model have been proposed. For example, the category $CPM(\mathbf{C})$, which describes mixed states, and the category $CP^*(\mathbf{C})$,categories, which is the category of Frobenius algebras in a monoidal category. All of these variations on the work of Abramsky and Coecke are monoidal categories, but they all have a different approach to classical and quantum information.

Simultaneously, symmetric monoidal 2-categories were proposed by Vicary in [20], as a model for general physical theories. The motivation was to give a generalisation of the 2-category $2Hilb$, which makes it possible to reason about quantum protocols as a whole, independent of branching due to measurement.

1.2 Classical and Quantum information

With classical information, we mean information that can be processed by classical computers. This is usually modelled by collections of bits, and operations on bits. Bits are two-valued systems. We call the different values of a system the states of the system.

Quantum information is given by Hilbert spaces. The elements of these Hilbert spaces, up to a complex scalar multiple, represent the set of all possible states. Information about the state of an object is obtained by measurement, which is given by self adjoint operators. These self adjoint operators are sums of orthogonal projectors $\{P_i\}$, which sum to the identity.

One of the distinguishing properties of quantum theory is that a measurement alters the state of the system that it measures. The possible outcomes of a measurement on a state

ϕ, are given by the images $P_i(\phi)$ of ϕ, under the projectors. After the measurement, the state is either destroyed, which we call a demolition measurements, or altered to the state of the outcome of the measurement $P_i(\phi)$, which we call a non demolition measurement. So we cannot obtain information about the original state by measurement, but we do know the state that the system is in after the measurement.

If we apply the same measurement twice, the second time the system behaves classically in the sense that we obtain information about the state, but it is not altered. This is due to self adjointness of the measurement. Measurements are generally not reversible because we do not know the original states. As well as measurements, there are also reversible operations.

Quantum systems have the property that parts of the system can interact with each other, even when they are spatially separated. We call this entanglement of quantum systems. We can no longer accurately describe the systems individually. Together they form a compound system, which is the tensor product of the individual systems. If we apply a measurement on part of a compound system, this instantaneously determines the state of the other part. This property is the key to interesting quantum protocols such as quantum teleportation, dense coding and quantum key distribution.

The counterpart of the bit is the qubit: the two dimensional complex Hilbert space $\mathbb{C}^2$, up to complex scalar multiples. The possible states of a qubit are given by all non zero, normalised elements of $\mathbb{C}^2$: normalised 2-dimensional complex vectors. Any orthonormal basis of $\mathbb{C}^2$ is a pair of 2-dimensional vectors, so this corresponds to a bit. Therefore, we can encode classical information as the basis of a quantum system. An orthonormal basis defines a family of orthogonal projectors that project to the basis vectors, therefore it corresponds to a measurement. The space of all possible outcomes of this measurement is exactly the corresponding bit.

Similarly, if a quantum system is in one of its base states, it behaves classically with respect to the measurement corresponding to this basis. In this case, the outcome of the measurement equals the original state with probability 1 when measured. This requires a specification of a state and of a measurement. Consequently, we cannot talk about classical states in general within the context of quantum information.

However, in many processes in quantum computing, the interaction between classical and quantum information plays an important role. The first example is a measurement, since it takes quantum data as input and classical data as output. Another important example is a controlled operation. This is a finite collection of operations $f = \{f_i\}_{i=1,\dots,n}$ on quantum systems. The input has a classical component that can take the values $1, \dots, n$ and a quantum component Q. The controlled operation $f(i, Q)$ gives $f_i(Q)$ as output. These two examples of interaction of classical and quantum information are, alongside entanglement and reversible operations, the building blocks of quantum teleportation.

There are different paradigms for quantum computing, such as the circuit model and measurement-based quantum computing. In all of them, the interaction between classical information and quantum information plays a fundamental role. Naturally, the semantics for these types of quantum computing should involve a model that includes both quantum information and classical information.

1.3 Physical processes and interpretations of Quantum Theory

'Physical processes' is the general term for all things that can happen to a physical object, like a qubit. In classical physics, all physical processes are predictable, and in theory, reversible. In quantum mechanics we know from experience that measuring a system changes the system in an uncontrolled way. When we measure a system, the outcome of the measurement and the resulting system is in one of the base states. We do not know in advance which of the base states. Even if we do know in which state the quantum system is we can only calculate the probability that it will be in one of the base states after the measurement. As a result, we cannot obtain information about a system without changing its state. This change is unpredictable, and irreversible. So in quantum mechanics, next to ordinary physical processes, there is another fundamentally different type of physical process.

One interpretation of this phenomenon is called the collapse of the wave function. Quantum systems are described as wave functions which describe all possible states. When measured, these possible states are reduced to the one that corresponds to the measurement outcome.

Within the Hilbert space model for quantum theory, these two types of physical processes are exactly captured by density matrices and morphisms between density matrices, which are called completely positive maps.

Within this framework, we can represent mixtures of classical and quantum systems, such as indexed sets of quantum systems. These mixtures arise naturally in physics, for instance when we have statistical information about a collection of quantum systems. Similarly, this is the case when we have a lack of knowledge about a system, because we can deal with this by assigning a probability distribution on all possible states. Both cases can be described by an indexed collection of quantum states, such that each state has a certain probability. The index set is a system of classical information, while the individual systems are quantum systems. This framework also captures systems that are entangled to another system we do not have knowledge about, or badly isolated systems that are altered because of an interaction with their environment. In this case, the environment can be regarded as a quantum system that is entangled to the system we are looking at.

The difficulty with this interpretation is that we cannot exactly describe when a system collapses and when we are dealing with an ordinary unitary operation. By conducting experiments, we know that relatively bigger quantum systems are less stable then smaller quantum systems; in other words, they are more likely to collapse into a classical system by themselves. There is no answer to the question of what the cause of the collapse is. Furthermore, it is difficult to unify this interpretation with special relativity theory for the following reason: If we have two entangled systems and one is being measured, then the other one should collapse simultaneously. However, in general relativity theory, 'simultaneously' has no meaning, as there exists no notion of 'the same time' for objects that are far apart. So there is no answer to the question of when exactly this wave function collapses.

Alternatively, we can describe a measurement as a physical operation that causes an entanglement between the system that is being measured and the bigger system around it, including the person who is doing the experiment. This can be seen as a flow of information between the system and its environment. In that case, the measurement outcome that is observed by the experimenter corresponds to the state that the quantum system is in, but the pair of the state of the quantum system and the state of the measurement outcome are in a superposition of all possible states with corresponding outcomes. Let us say that we have a quantum system X that is in superposition $ax_1 + bx_2$ of the base states x_1 and x_2. If this is being measured by an experimenter, a new system is created: the experiment outcome Y that has y_1 and y_2 as possible outcomes (these correspond to x_1 and x_2 respectively). The measurement of X causes an entanglement between X and Y. The resulting joint system after the measurement is then in the superposition $ax_1y_1 + bx_2y_2$ of x_1y_1 and x_2y_2, and there is no collapse.

This interpretation does not correspond to our experience, as we experience only one state. At the same time, it is not very clear whether we could know if we are in a superposition of spaces. This raises the question if and how this interpretation can possibly be falsified. It is difficult to argue that this interpretation is impossible, as we do not know of any fundamental difference between very small systems, which can be in a superposition, and larger systems which includes the experimenter. From this perspective, the distinguishing factor between quantum systems and classical systems is rather isolation than size. The only reason why the measurement of bigger objects does not cause a collapse can be that the experimenter is already entangled to the object that is measured. We only experience quantum behaviour in very small systems because these systems are easier to isolate. From this perspective, quantum behaviour is something relative, instead of something absolute.

The paragraph above is a very brief description of what is known by the 'many worlds' interpretation or 'decoherence' interpretation of quantum mechanics. 'Decoherence' is the term for the information flow of quantum systems to their environment, and 'many worlds' refers to the idea that everything in our world is in a superposition of many possible worlds.

There is a mathematical model that corresponds to both interpretations, this is the category $2Hilb$. This captures only pure quantum systems, but from an 'outside' perspective that takes the environment of the system into account.

Chapter 2

Symmetric monoidal dagger compact closed categories

2.1 Definitions

Category theory is an elegant tool to describe physical processes abstractly. The objects of a category represent the physical systems, and the morphisms represent the evolution of the state of a system. We recall the definition of a category:

Definition 2.1.1. A category $\mathbf{C}$ is a structure that consists of

- a collection of objects
- collections of morphisms $\mathbf{C}(A, B)$ for all objects A,B, which are called hom-sets

Such that

- For all morphisms $f \in \mathbf{C}(A, B)$, $g \in \mathbf{C}(B, C)$, we have the composition $g \circ f \in \mathbf{C}(A, C)$
- For every object A, there is an identity morphism id_A
- For all morphisms f,g,h, such that the composition $h \circ g$ and $g \circ f$ is well defined, $(h \circ g) \circ f = h \circ (g \circ f)$

2.1.1 Monoidal categories

In reality, we may have several physical systems that we want to describe in parallel. This gives the motivation to work with monoidal categories.

Definition 2.1.2. Monoidal category

A monoidal category is a category **C** equipped with the following data:

- a tensor product $\otimes$, which is a functor from $\mathbf{C} \times \mathbf{C}$ to **C**
- an object I, which we call the unit object
- a natural isomorphism whose components, which correspond to triples of objects of **C**, are morphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C) \to A \otimes (B \otimes C)$. These components are called the associators.
- natural isomorphisms λ and ρ whose components, which correspond to objects of **C**, are the morphisms $I \otimes A \xrightarrow{\lambda_A} A$ and $A \otimes I \xrightarrow{\rho_A} A$, respectively. These components are called left- and right unitors respectively.

We require that every well-formed equation built from $\circ$, $\otimes$, id, α, α^{-1}, λ, λ^{-1}, ρ and ρ^{-1} is satisfied. In that case we say that **C** satisfies the coherence condition.

We can think of the tensor product of different systems as systems that are spatially separated.

There are several examples of monoidal categories, the following three are described in [9]:

Example 2.1.3. *FHilb*, the category of finite dimensional Hilbert spaces and linear maps. The tensor product is the ordinary tensor product of Hilbert spaces, the unit object is the one dimensional Hilbert space $\mathbb{C}$, the dagger is the adjoint of matrices. The associator and left and right unitors are the unique linear maps $(A \otimes B) \otimes C \xrightarrow{\alpha} A \otimes (B \otimes C)$, $I \otimes A \xrightarrow{\lambda} A$ and $A \otimes I \xrightarrow{\rho} A$ such that $\alpha((a, b), c)) = (a, (b, c))$, $\lambda(1, a) = a$ and $\rho(a, 1) = a$.

Example 2.1.4. Set, the category of sets and functions. The tensor is given by the Cartesian product, the unit object is the one element set $\{*\}$. The associator is the function $\alpha_{A,B,C}((a, b), c)) = (a, (b, c)$ on all elements $a \in A, b \in B, c \in C$. The left and right unitors are the functions $\lambda(*, a) = a$ and $\rho(a, *) = a$ for all $a \in A$.

Example 2.1.5. *Rel*, the category of sets and relations. This is a monoidal category in exactly the same way as Set is.

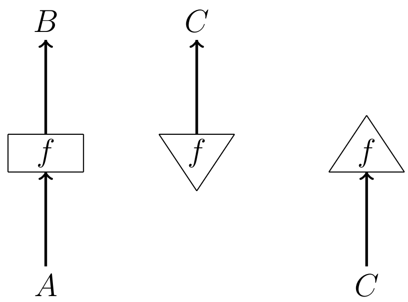
Definition 2.1.6. The states of an object A are defined as the morphisms $I \to A$. If A is a tensor product $A_1 \otimes A_2$, the entangled states are the morphisms $I \xrightarrow{f} A_1 \otimes A_2$ that are not product states. This means that they cannot be written as $I \xrightarrow{\lambda^{-1}} (I \otimes I) \xrightarrow{f_{A_1} \otimes f_{A_2}} (A_1 \otimes A_2)$.

The states of a Hilbert space H were defined as the elements of H, which are vectors. Every vector v corresponds to the linear map from $\mathbb{C}$ to H that maps 1 to v. This corresponds to the abstract definition of a state.

Monoidal categories have an elegant graphical language, which is sound and complete by [16]. In this language, the unit object I is written as an empty space, an object A is written as the wire

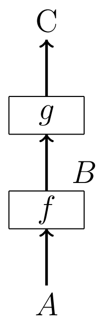
(2.1)

This is also the picture of the identity morphism id_A on A. A morphism $A \xrightarrow{f} B$ is written as a box and morphisms $I \xrightarrow{g} C$, $C \xrightarrow{I}$ are written as triangles.



(2.2)

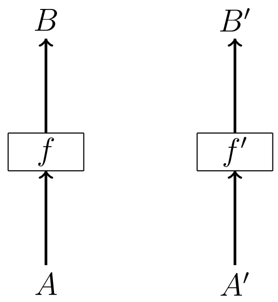
The composition of morphisms $g \circ f$ is written as



(2.3)

For the interpretation of the diagrams, time flows upwards, so the bottom of the diagram is the input and the top of the diagram is the output. There is always an input and an output, because when the picture at the bottom or top is empty, this is the unit object.

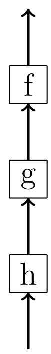
The tensor product of morphisms $f \otimes f'$ is written as



(2.4)

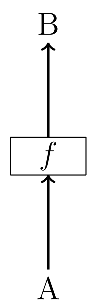
Space is presented horizontally: a juxtaposition of the objects normally means that the objects are spatially separated.

Associativity of composition and the identity morphism are implicit in the diagrammatic language because $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are both expressed by



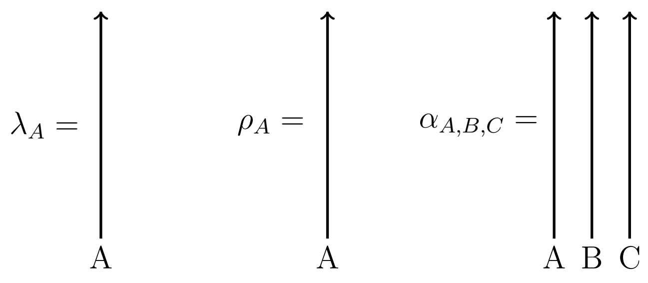
(2.5)

and $f \circ \mathrm{id}_A$ and $\mathrm{id}_B \circ f$ are both expressed by



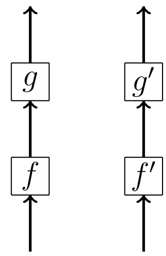
(2.6)

The length of the wires is not important. λ_A, ρ_A and the associator $\alpha_{A,B,C}$ are given by



(2.7)

The interchange law $(g \circ f) \otimes (g' \circ f') = (g \otimes g') \circ (f \otimes f')$ which follows from functoriality of $\otimes$ is also immediately clear in our diagrammatic language as both sides of the equality are represented as

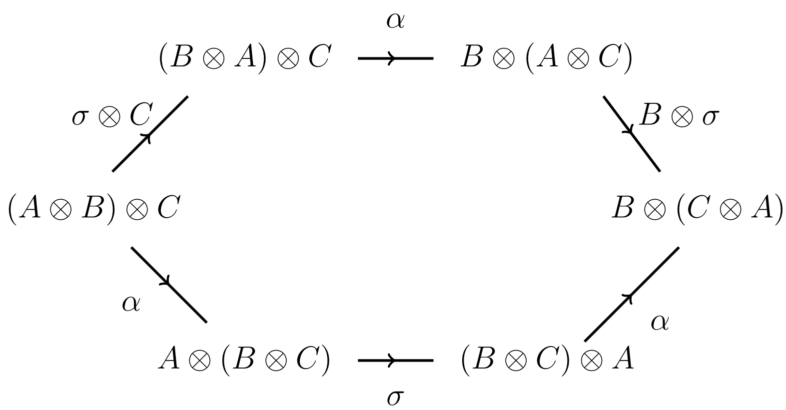


(2.8)

2.1.2 Symmetry

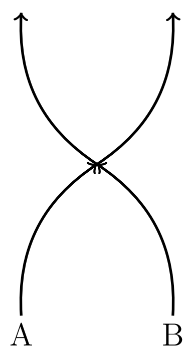
If we interpret the tensor product as spatial separation of systems, it is reasonable to argue that $A \otimes B$ is more or less the same as $B \otimes A$. This is made precise by the following definition.

Definition 2.1.7. A monoidal category **C** is symmetric if there exists a natural isomorphism σ consisting of morphisms $A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$, such that $\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A \otimes B}$ and $\rho_A = \lambda_A \circ \sigma_{A,I}$ for all objects A,B of **C**, such that the diagram below commutes.



(2.9)

In our diagrammatic language, we draw the map $\sigma_{A,B}$ as



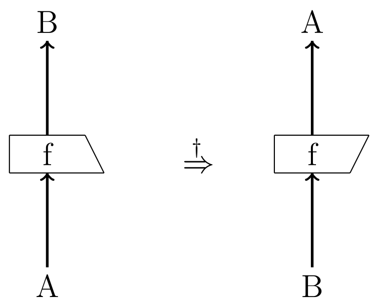
(2.10)

2.1.3 Dagger

Symmetric monoidal categories do not capture an analogue of the inner product of Hilbert spaces. This is a defining property of Hilbert spaces, which is used to formulate the Born rule. The Born rule enables us to determine the probability distribution on the possible outcomes of a measurement. The inner product corresponds to the notion of adjoints, which can be generalised by a dagger functor.

Definition 2.1.8. A dagger functor on a category $\mathbf{C}$ is a contravariant functor $\mathbf{C} \overset{\dagger}{\Rightarrow} \mathbf{C}$ that is the identity on objects satisfies $\dagger(\dagger(f)) = f$ for all morphisms f. We write $f^\dagger$ for $\dagger(f)$. A unitary morphism is a morphism u such that $u^\dagger = u^{-1}$. A symmetric monoidal dagger category is a symmetric monoidal category with a dagger functor such that α, σ, ρ and λ are unitary and $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$.

We depict the image of the dagger functor in our graphical language as the original graph flipped about a horizontal axis. To make this clearly visible, we will make our boxes asymmetric in the horizontal axis. Graphically it looks like

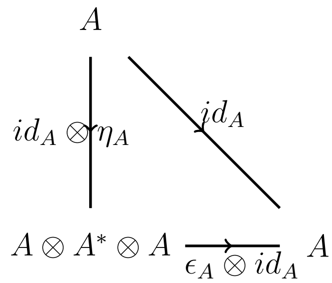


(2.11)

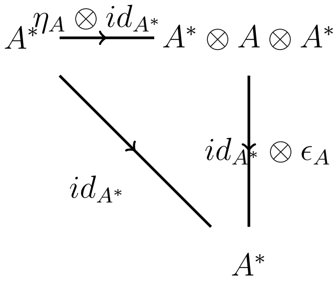
2.1.4 Compactness

In quantum theory, there are different levels of entanglement with an upper bound, which we call a maximally entangled state. This is formalized in terms of dual objects.

Definition 2.1.9. An object A in a monoidal category is left dual to an object A^* (and A^* is right dual to A) if there exist two morphisms $I \xrightarrow{\eta_A} A^* \otimes A$ and $A \otimes A^* \xrightarrow{\epsilon_A} I$ such that the following two diagrams commute. We call these diagrams the snake equations. The snake equations are a generalisation of the line-yank equations, which are defined in terms of linear maps on Hilbert spaces.



(2.12)



(2.13)

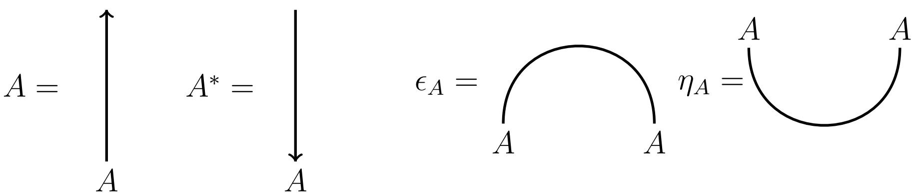
In a monoidal dagger category and in a symmetric monoidal category, left-dual objects are are the same time right-dual objects, witnessed by $\eta^\dagger$ and $\epsilon^\dagger$ and by $\sigma \circ \eta$ and $\epsilon \circ \sigma$ respectively. This was observed in [19] and [9].

Definition 2.1.10. A symmetric monoidal dagger category **C** is compact if for every object A there is a dual object A^*. We call a category dagger compact, when it is symmetric monoidal dagger category that is compact and $\epsilon^\dagger = \sigma \circ \eta$ and $\epsilon \circ \sigma = \eta^\dagger$.

Example 2.1.11. *FHilb* is a dagger compact category. The symmetry isomorphism $\sigma_{A,B}$ is given by the unique linear map $\sigma : A \otimes B \to B \otimes A$ given by $\sigma(a, b) = (b, a)$. The dual of a Hilbert space H is the Hilbert space H^*, of continuous linear functions from H to $\mathbb{C}$. Let $\{x_i\}$ be an orthonormal basis of a Hilbert space H. The morphism $\eta_H : \mathbb{C} \to H^* \otimes H$ is the unique linear map such that $\eta_H(1) = x_i^* \otimes x_i$, where $\{x_i\}$ is the collection of states that corresponds to the orthonormal basis of H. $\epsilon_H : H \otimes H^* \to \mathbb{C}$ is defined as $\epsilon_H(x_i, x_j^*) = \langle x_i | x_j \rangle$ on the basis $\{(x_i, x_j^*)\}$ of $H \otimes H^*$. The dagger functor † maps linear maps to their adjoint.

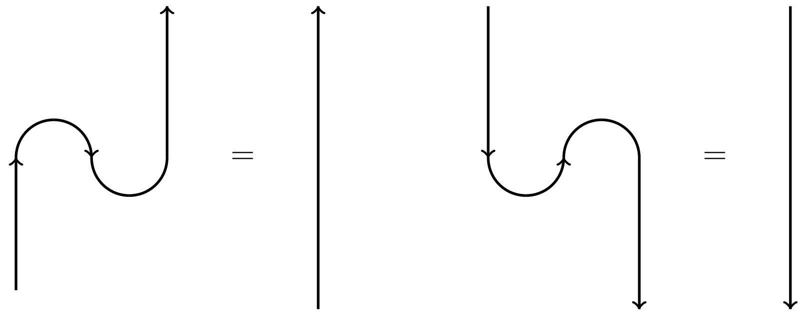
Example 2.1.12. *Rel* is also a dagger compact category. The symmetry morphism $\sigma : A \otimes B \to B \otimes A$ is given by the relation $(a, b) \sim (b, a)$. Every object is its own dual and the morphisms $\eta : \{*\} \to A \times A$ and $\epsilon : A \times A \to \{*\}$ are given by the relations $\{* \sim (a, a) | a \in A\}$ and $\{(a, a) \sim * | a \in A\}$. The dagger : $B \xrightarrow{R^\dagger} A$ of a relation $A \xrightarrow{R} B$ is defined as $\{bR^\dagger a | a \in A, b \in B, aRb\}$.

Graphically we write dual objects with reversed arrows, and witnesses of duals as caps and cups. The direction of the arrow is only relative.



(2.14)

The snake equations become:

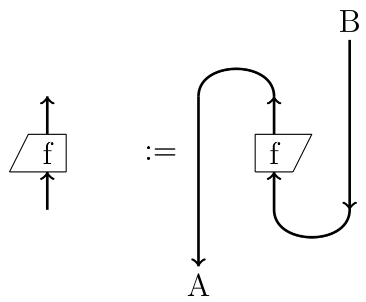


(2.15)

Compactness is a useful condition because in our diagrammatic language this allows us to bend wires of any diagram without changing its meaning, as long as the starting and end points of a wire stay in the same place.

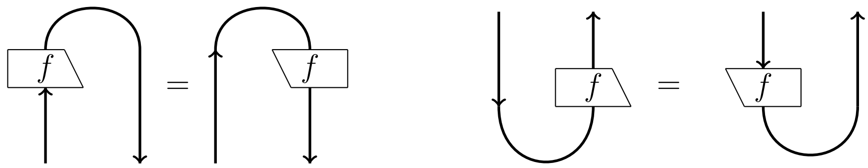
In a compact category, duality of objects extends to a duality functor, that maps every object and every morphism to their dual. This is proven in [9]. The dual of a morphism is defined as follows:

Definition 2.1.13. In a compact monoidal category, the dual $A^* \xrightarrow{f^*} B^*$ of the morphism $A \xrightarrow{f} B$ is defined as



(2.16)

It follows from the definition that



(2.17)

Note that in mathematical literature f^* is often used for the adjoint of a matrix, which in this setting corresponds to $f^\dagger$.

Duality of objects is independent of the choice of the morphisms η and ϵ, which is shown in [9]. The states η correspond to generalised Bell states in Hilbert space. These states are called maximally entangled. If we measure one part of a maximally entangled state, this completely determines the other part. This follows directly from the equation above.

By lemma 3.16 of [9], the dagger functor commutes with the duals functor. The composition $f_* := f^{\dagger *} = f^{*\dagger}$ is called the conjugation of f. In $FHilb$, this corresponds to the conjugation of matrices.

The diagrammatic language of dagger compact categories that we described below is complete and sound. This is proven in [16].

2.1.5 Superposition

A category $\mathbf{C}$ is enriched in commutative monoids, when each hom-set $\mathbf{C}(A, B)$ has a commutative monoid structure that is compatible with composition. This means that there is an action $'+'$ on $\mathbf{C}(A, B)$ and a unit element $u \in \mathbf{C}(A, B)$ for all A, B, such that

$$f + g = g + f \tag{2.18}$$

$$f + (g + h) = (f + g) + h \tag{2.19}$$

$$f + u = f \tag{2.20}$$

$$(g + g') \circ f = (g \circ f) + (g' \circ f) \tag{2.21}$$

$$g \circ (f + f') = (g \circ f) + (g \circ f') \tag{2.22}$$

This is also known as a superposition rule, as it is necessary to be able to express superposition of states.

2.1.6 Dagger biproducts

Recall that in a category $\mathbf{C}$ that is enriched in commutative monoids, a biproduct of two objects A_1, A_2 is an object $A_1 \oplus A_2$, together with morphisms $i_j : A_j \to A_1 \oplus A_2$ and $p_j : A_1 \oplus A_2 \to A_j$ for $j = 1, 2$, such that

$$\mathrm{id}_{A_j} = p_j \circ i_j \tag{2.23}$$

$$0_{A_1,A_2} = p_{A_2} \circ i_{A_1} \tag{2.24}$$

$$0_{A_2,A_1} = p_{A_1} \circ i_{A_2} \tag{2.25}$$

$$\mathrm{id}_{A_1 \oplus A_2} = (i_{A_1} \circ p_{A_1} + i_{A_2} \circ p_{A_2}) \tag{2.26}$$

Definition 2.1.14. In a dagger category $\mathbf{C}$, a dagger biproduct of objects A_1, A_2 is a biproduct such that $i_j^\dagger = p_j$ for $j = 1, 2$.

$FHilb$ and Rel have dagger biproducts. In $FHilb$, they are given by the direct sum of Hilbert spaces and in Rel they are given by the disjoint union of sets.

By lemma 3.21 of [9], if a compact category has biproducts, then the tensor product distributes over the biproduct.

2.2 Properties of dagger compact categories

As we have seen already, states can be defined abstractly, within the framework of symmetric monoidal dagger compact categories. It turns out that many properties of Hilbert spaces can be generalised in this model.

2.2.1 Scalars

Definition 2.2.1. In a monoidal category $\mathbb{C}$, the scalars are the morphisms in $\mathbb{C}(I, I)$. By lemma 2.2 of [9], scalars are commutative in any monoidal category. Let $I \xrightarrow{a} I$ be a scalar, and $A \xrightarrow{f} B$ be a morphism, scalar multiplication $a \cdot f$ is given by the morphism that makes the diagram below commute:

$$\begin{array}{ccc} \text{A} & \xrightarrow{a \cdot f} & \text{B} \\ \Big\downarrow {\scriptstyle \lambda_A^{-1}} & & \Big\downarrow {\scriptstyle \lambda_B} \\ I \otimes A & \xrightarrow[a \otimes f]{} & I \otimes B \end{array} \tag{2.27}$$

By lemma 2.4 of [9], this satisfies the properties of scalar multiplication: $i \cdot f = f$, $a \cdot (b \cdot f) = (a \circ b) \cdot f$, $(a \cdot f) \circ (b \cdot g) = (a \circ b) \cdot (f \circ g)$ and $(a \cdot f) \otimes (b \cdot g) = (a \circ b) \cdot (f \otimes g)$.

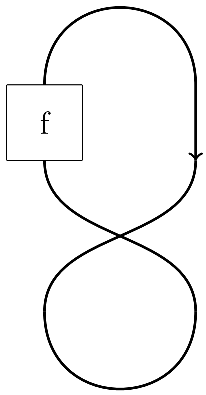
2.2.2 Inner product

One of the defining properties of a Hilbert space is the inner product. We can generalize this concept using the dagger.

Definition 2.2.2. The abstract inner product on a monoidal dagger category is an operation $\langle -, - \rangle$ that assigns a scalar to a pair of states (ϕ, ψ) of an object A. This operation is given by $\langle \phi, \psi \rangle = \phi^\dagger \circ \psi$. In $FHilb$, this corresponds to the inner product, since it equals $\langle 1, \phi^\dagger \circ \psi \rangle = \langle \phi, \psi \rangle$.

2.2.3 Trace

Definition 2.2.3. For any morphism $A \xrightarrow{f} A$ in a dagger compact category, the abstract trace is defined as $\epsilon_A \circ (f \otimes \mathrm{id}_A) \circ \sigma_{A^*,A} \circ \eta_A$. Graphically this is given by



(2.28)

If we apply this to a morphism $H \xrightarrow{f} H$ of the category $FHilb$, we get the scalar $\sum_{i,j} f_{i,j} \langle x_j | x_i \rangle = \sum_i f_{i,i}$, where $\{x_i\}$ is the basis of H and $f_{i,j}$ are the scalars such that $f(x_i) = \sum_j f_{i,j} x_j$. This is exactly the trace of the matrix f.

2.2.4 Classical structures

As we argued in the introduction, any semantics for quantum computation needs to involve the notion of classical information. In the category $FHilb$, any orthogonal basis of a Hilbert space corresponds to classical information. This classical information equals the set of all possible outcomes of the measurement corresponding to this basis. Measurements are self-adjoint operators. By the spectral theorem, these are sums of projectors.

Bases can be generalised in monoidal categories with dagger biproducts as follows:

Definition 2.2.4. Let **C** be a monoidal category with biproducts. An abstract basis of an object A is a unitary morphism $base : I_1 \oplus ... \oplus I_n \to A$. The basis elements are the morphisms $I \xrightarrow{i_j} (I_1 \oplus ... \oplus I_n) \xrightarrow{base} A$ for $j = 1, ..., n$.

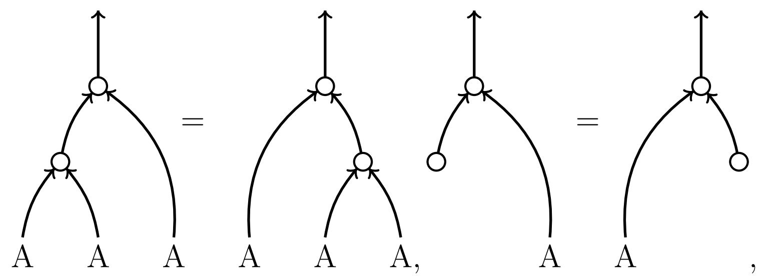
An abstract basis defines an isomorphism between an object A and a biproduct of unit objects $I \oplus ... \oplus I$. Therefore, we can represent classical information as biproducts of the unit object. This generalises the fact that classical systems are completely determined by their individual components.

Every dagger biproduct defines a measurement, given by $\sum_n i_n \circ p_n$. By the definition of a biproduct, this is the sum of projections $P_n := i_n \circ p_n$ and $\sum_n P_n = \mathrm{id}$. Biproducts also play a role in the characterisation of quantum states. By lemma 2.10 of [9], if a category has biproducts, it has a unique superposition rule.

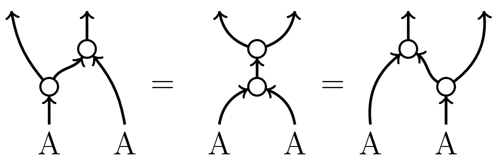
The drawback of using biproducts to characterise classical information is that there is no diagrammatic language to express biproducts, so this complicates things for practical purposes.

Another way to look at classical information within monoidal categories is by exploiting the fact that classical information can be copied and deleted, while there is no uniform way to copy and delete quantum systems. This is known as the no-cloning and no-deleting theorems. A copy map on a Hilbert space $H \xrightarrow{copy} H \otimes H$ is a linear map which is defined on a basis $\{x_i\}_i$ as $x_i \xrightarrow{copy} x_i \otimes x_i$. The image of an element $\phi = c \cdot x_i$ under this map is $c \cdot (x_i \otimes x_i)$. This equals $\phi \otimes \phi$ when is $c = 1$, hence only the basis elements $\{x_i\}$ are copied. Coecke and Pavlovic generalised this property of classical states in [4]. They characterised classical states as those states that can be copied by a commutative dagger Frobenius algebra, which is the structure that abstractly describes a copy map.

Definition 2.2.5. Let **C** be a monoidal dagger category. A dagger Frobenius algebra on an object A is given by a pair of morphisms $\triangledown_A : A \otimes A \to A$ and $u_A : I \to A$, such that $\triangledown_A$ and u_A satisfy the equations below.

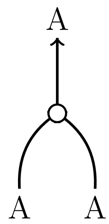


(2.29)



(2.30)

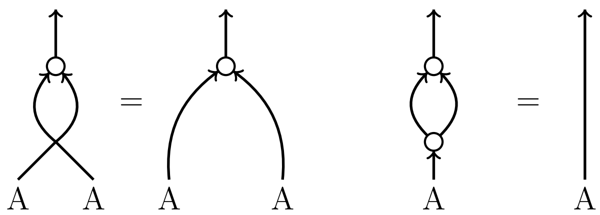
We have used the following diagrammatic notation for u_A and $\triangledown$:

$A \otimes A \xrightarrow{\bigtriangledown} A$ =:  $I \xrightarrow{u_A} A$ =: 

(2.31)

Note that by the first two equations, the maps $\bigtriangledown$ and u_A form an internal monoid and their daggers form an internal comonoid.

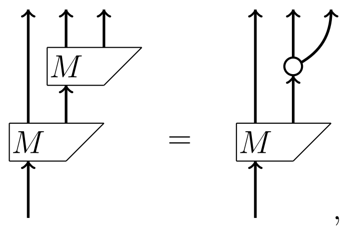
The two equations below define commutativity and specialness of dagger Frobenius algebras, respectively.



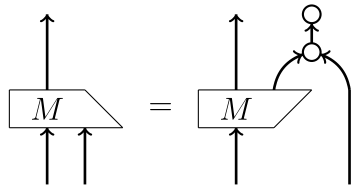
(2.32)

In [19] it is shown that special commutative dagger Frobenius algebras on objects in $FHilb$ do indeed correspond to orthogonal bases of Hilbert spaces.

Measurements relative to a classical structure can be defined abstractly as morphisms $A \xrightarrow{M} A \otimes A$ such that

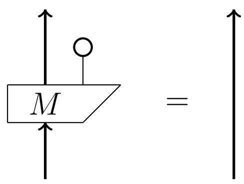


, (2.33)



(2.34)

And a non demolition measurement also satisfies



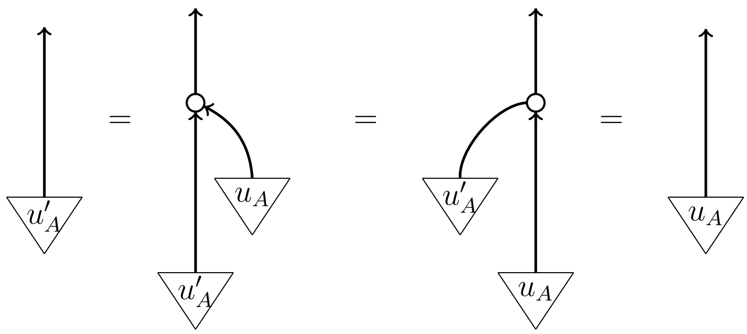
(2.35)

The left copy of the state A in the output of a measurement should be interpreted as the resulting quantum system, while the right copy of A corresponds to the classical measurement outcome.

Note that the dagger of the copy map Δ, satisfies all three equations, so it defines a non demolition measurement.

Any biproduct with a basis induces a dagger Frobenius algebra, which is the map that copies the basis elements. This is special and commutative by the definition of the biproduct. Dagger Frobenius algebras are a more general notion of classicality, as they do not depend on the existence of biproducts.

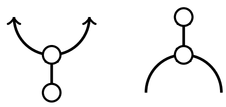
Any dagger Frobenius algebra is uniquely determined by its multiplication. This is easy to see because when $(\bigtriangledown_A, u_A)$ and $(\bigtriangledown_A, u'_A)$ both define a dagger Frobenius algebra on an object A, then we have



, (2.36)

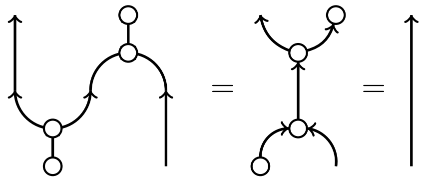
by the Frobenius laws. In this picture we represent $\bigtriangledown_A$ by the white dot.

Dagger Frobenius algebras induce a self duality on objects, given by the following maps



(2.37)

The snake duality follows from the Frobenius rules:



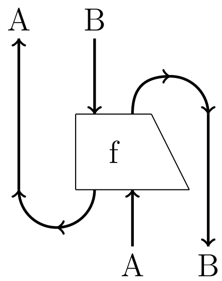
(2.38)

2.3 Completely positive operators

Dagger compact closed categories do not capture mixed states. Historically, Von Neumann argued that quantum systems should be described as completely positive operators because these capture both quantum features and classical probabilistic features. Generalising this idea, Selinger proposed the category $CPM(\mathbf{C})$ in [17] as the framework for semantics of quantum computing.

Definition 2.3.1. A positive map $A \xrightarrow{f} A$ is a map of the form $A \xrightarrow{g} B \xrightarrow{g^\dagger} A$ for some pair (B,g).

Definition 2.3.2. A completely positive map is a map $A^* \otimes A \xrightarrow{f} B^* \otimes B$ such that



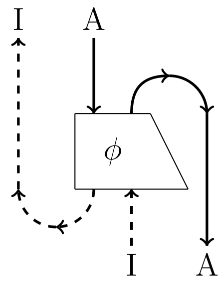
(2.39)

is a positive map.

Definition 2.3.3. The objects of $CPM(\mathbf{C})$ are the same as the objects of $\mathbf{C}$, and the morphisms in $CPM(\mathbf{C})(A, B)$ are completely positive maps $A^* \otimes A \xrightarrow{f} B^* \otimes B$.

By [17], the identity morphism is completely positive, and completely positive maps are closed under composition, so $CPM(\mathbf{C})$ is a category. Furthermore, if $\mathbf{C}$ is dagger compact, then $CPM(\mathbf{C})$ is dagger compact.

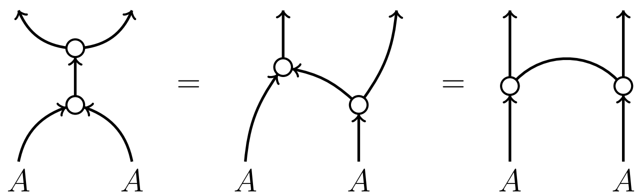
States of $CPM(\mathbf{C})$), which are maps $I^* \otimes I \xrightarrow{\phi} A^* \otimes A$, correspond to positive maps from A to A. This is shown in the picture below. Completely positive maps are exactly the morphisms that preserve this correspondence, since they are closed under composition.



(2.40)

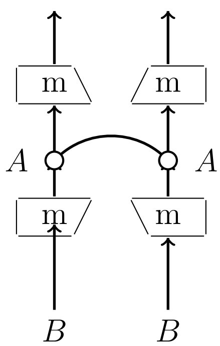
CPM(**FHilb**) corresponds to the category of density matrices and completely positive operators. There is a one to one correspondence between positive linear maps, and density matrices. Completely positive operators are exactly those operators that preserve density matrices.

Given a classical object A with a commutative special dagger Frobenius algebra, we can define the completely positive map



(2.41)

By specialness, this map is idempotent, and by the Frobenius laws it is self-adjoint, so this map is a measurement. So far we have only discussed a special form of measurement, namely non-degenerate measurements. In Hilbert space, a non-degenerate measurement on a Hilbert space H of dimension k is a collection of k orthogonal projections P_i such that the images of the projections are 1-dimensional and span H. Sums of orthogonal projections of arbitrary dimension are called degenerate measurements. If we characterise classicality using biproducts, the analogue of a degenerate measurement is a collection of projections onto the summands of an arbitrary biproduct. In terms of Frobenius algebras, a general measurement is defined as

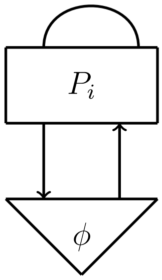


(2.42)

This measurement is non-degenerate if m is unitary. This was observed in [4].

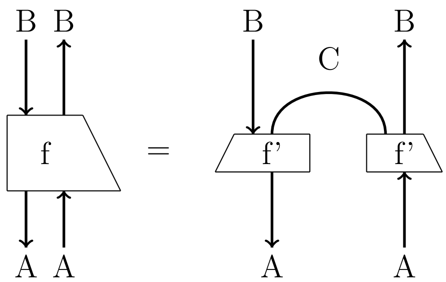
If **C** has biproducts, we can calculate the probability of getting a measurement outcome $P_i(\phi)$ when applying a measurement $\sum_i P_i$.

Definition 2.3.4. The probability of getting the outcome corresponding to the projector P_i when measuring a state ϕ is given by



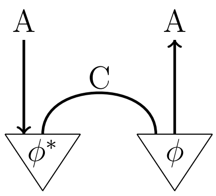
(2.43)

In [17] it is shown that a map f is completely positive if and only if there exists an object C and a morphism $A \xrightarrow{f'} C \otimes B$ such that



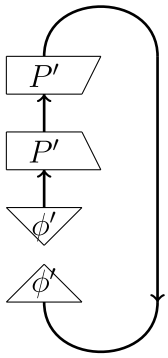
(2.44)

As a consequence, states of $CPM(\mathbf{C})$ are of the form



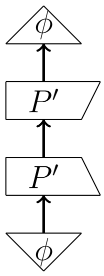
(2.45)

It follows that the probability, given by the abstract Born rule, corresponds to taking the following trace



(2.46)

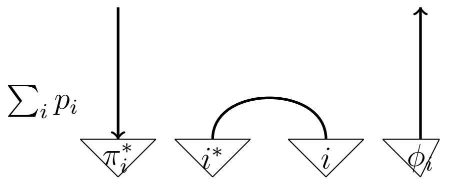
This corresponds by compactness to



, (2.47)

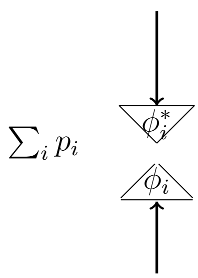
which is exactly the Born rule as it was defined on density matrices.

$CPM(\mathbf{C})$ captures the different notions of mixed states as we discussed in the introduction. We can perform operations on states of the system A, which is part of the bigger system $C \otimes A$. We can also represent statistical ensembles of quantum states $\{\phi_i\}$ with probabilities p_i, as:



(2.48)

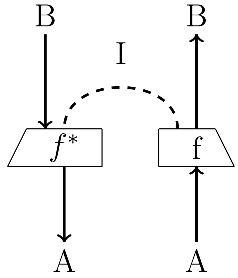
Where the states $|i>$ form an orthonormal basis of A. In Hilbert space, this corresponds to the density matrix



(2.49)

This can be understood as a mix of quantum amplitudes which are inherent to the states ϕ_i and classical probabilities p_i.

We can embed the category $\mathbf{C}$ in $CPM(\mathbf{C})$ with the functor $F_{\mathbf{C},CPM(\mathbf{C})}$. This functor is the identity on objects, and on morphisms it is defined as

$$F_{\mathbf{C},CP(\mathbf{C})}(f) = f^* \otimes f =$$ 

(2.50)

This functor is almost faithful because for morphisms $f, g : A \to B$, $F_{\mathbf{C},CPM(\mathbf{C})}(f) = F_{\mathbf{C},CPM(\mathbf{C})}(g)$, if and only if there exists a unit scalar ϕ, such that $f = \phi \cdot g$. This is shown in [9]. We call a state pure if it is in the image of $F_{\mathbf{C},CPM(\mathbf{C})}$. In that case the mixing is only by scalars, so it is trivial. We call all the other states mixed states.

Example 2.3.5. By [11], in the category *Rel*, positive maps are the relations that are symmetric and satisfy $x \sim y \Rightarrow x \sim x$. Completely positive maps are relations that satisfy $(a, a') \sim (b, b')$ if and only if $(a', a) \sim (b', b)$ and $(a, a) \sim (b, b)$.

Selinger shows in [17] that the category $CPM(\mathbf{C})$ of any monoidal dagger compact category $\mathbf{C}$ is again a monoidal dagger compact category.

2.4 Models for classical and quantum information

In dagger compact categories, we can characterise classical states using biproducts or commutative special Frobenius algebras. In both cases, the system only behaves classically with respect to a chosen basis or Frobenius algebra. Because classicality depends on a choice, there is no expression for classical systems in general. However, implicitly we assume that some systems are classical. For instance, in the abstract characterisation of a measurement, one of the output wires represents the measurement outcome, which is a classical system. To make this rigorous, we need another model that captures classical systems and quantum systems as distinct concepts. Several constructions of categories, based on dagger compact categories, have been proposed for this purpose.

2.4.1 $CP^{\oplus}(-)$

The biproduct completion of $CPM(\mathbf{C})$,was proposed by Selinger in [17] to formalise the interaction of classical and quantum systems.

Definition 2.4.1. The biproduct completion $\mathbf{C}^{\oplus}$ of a category $\mathbf{C}$ is the category where:

- The objects are finite tuples of objects of $\mathbf{C}$.
- The morphisms are matrices of morphisms of $\mathbf{C}$.
- Composition of morphisms is given by matrix multiplication and composition in $\mathbf{C}$.

Every category $\mathbf{C}$ that is enriched over commutative monoids has a biproduct completion by [13]. By [17], $CPM(\mathbf{C})$ is enriched over commutative monoids, so $CP^{\oplus}(\mathbf{C})$ is well defined as a category. By proposition 5.1 of [17], it is even a compact dagger category.

Note that if $\mathbf{C}$ has biproducts, that these are not preserved by the construction $CP^{\oplus}$, as morphisms between biproducts in this new category are given by matrices of completely positive morphisms, while morphisms between biproducts in $\mathbf{C}$ are matrices of any morphisms in $\mathbf{C}$.

We can embed the category $CPM(\mathbf{C})$ in $CP^{\oplus}(\mathbf{C})$ with the functor that maps objects A to the one-tuple $\langle A\rangle$ and morphisms f to the 1-by-1 matrix $\big(f\big)$. Note that this also defines a functor from $\mathbf{C}$ to $CP^{\oplus}(\mathbf{C})$, that maps the object A to the one-tuple $\rangle A\rangle$ and the morphisms f to the matrix $(f^* \otimes f)$.

In the category $CP^{\oplus}$, classical information is given by biproducts of the unit element $I \oplus ... \oplus I$ while purely quantum systems are those systems that are in the image of $\mathbf{C}$. Besides purely quantum systems and classical systems, there are also mixes of classical and quantum systems given by arbitrary biproducts. In $FHilb$, this corresponds to our notion of classical and quantum information: bits are of the form $\langle \mathbb{C}, \mathbb{C}\rangle$, which is the biproduct in $CP^{\oplus}(\mathbf{C})$, while qubits are of the form $\mathbb{C}^2$.

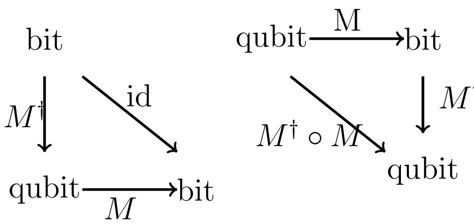
The category $CP^{\oplus}(\mathbf{C})$ captures both interpretations of quantum mechanics that were described in the introduction. If we restrict to the sub category $CPM(-)$, this describes systems involving a collapse, while if we restrict to biproducts of pure quantum systems, this describes systems according to the many worlds interpretation.

2.4.2 $\mathbf{Split}^{\dagger}(CPM(\mathbf{C}))$

As we have seen, measurements have classical, mixed or quantum systems as an output, depending on if they are degenerate or not. Another approach to characterise classical and quantum information is to identify measurements with their output. General measurements are dagger idempotent morphisms p, that is: $p = p \circ p = p^\dagger$. By proposition 5.8 of [18], dagger idempotents are uniquely determined by their image.

Definition 2.4.2. We say that a dagger idempotent $e : A \to A$ dagger splits, if there is an object B and a map $f : A \to B$, such that $e = f^\dagger \circ f$ and $\mathrm{id}_B = f \circ f^\dagger$.

Example 2.4.3. In *FHilb*, bits arise from dagger splittings of dagger idempotents on qubits. Recall that a bit corresponds to $\mathbb{C} \oplus \mathbb{C}$ and a qubit corresponds to $\mathbb{C}^2$. We obtain a bit from a qubit by the measurement $M \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a, d)$. We can encode a bit as the basis of a qubit with the adjoint of the measurement $M^\dagger(a, d) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ These maps form the dagger splitting:



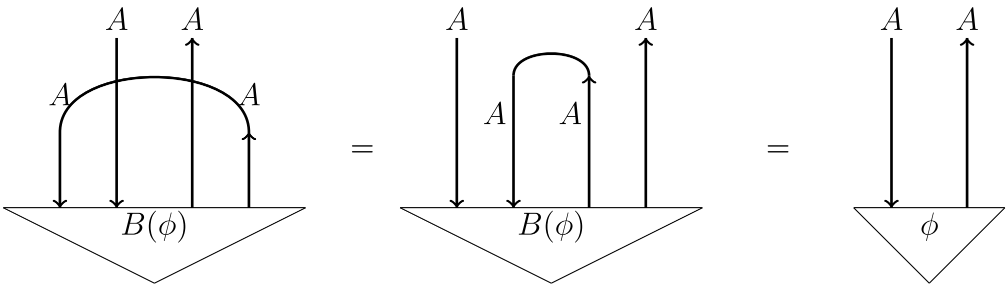
(2.51)

Definition 2.4.4. The category $\mathrm{Split}^\dagger(CPM(\mathbf{C}))$ is the category obtained by freely dagger splitting idempotents of $CPM(\mathbf{C})$. The objects are pairs (X,p), where X is an object of $CPM(\mathbf{C})$ and $p : X \to X$ is a dagger idempotent of $CPM(\mathbf{C})$. The morphisms $(X, p) \xrightarrow{f} (Y, q)$ are morphisms $X \xrightarrow{f} Y$ in $CPM(\mathbf{C})$ that satisfy $f = q \circ f \circ p$.

This construction is a restriction of the Karoubi envelope $Split(CPM(\mathbf{C}))$, to the class of dagger idempotents. By [18], if $\mathbf{C}$ is a monoidal dagger compact category with dagger biproducts, this also holds for $Split^\dagger(CPM(\mathbf{C}))$ There is a full embedding $CPM(\mathbf{C}) \hookrightarrow Split^\dagger(\mathbf{C})$ that maps objects A to (A, id_A) and which is the identity on morphisms. As $Split^\dagger(CPM(\mathbf{C}))$ has dagger biproducts, this embedding lifts to $CP^\oplus(\mathbf{C})$ by universality of the biproduct completion.

2.4.3 $CP^*(\mathbf{C})$

As we have seen in 2.2.4, pure classical states can be characterised as copyable states. However, this does not extend to classical mixed states. By the no-cloning theorem for classical probability, states of probability distributions on classical systems are not copyable either. However, we can characterise classical states in CPM as those states that can be broadcasted. A broadcast morphism is a Morphism $B : A \to A \otimes A$, such that for any state ϕ of A, we have

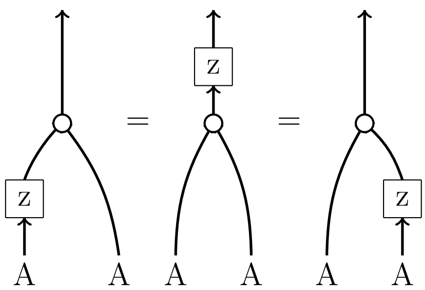


(2.52)

In [6], it is proven that the multiplication of a special Frobenius algebra defines a broadcasting map on the unit, but that this is only a completely positive morphism when the Frobenius algebra is commutative. In Hilbert spaces, commutative dagger Frobenius algebras correspond exactly to classical states. Special Frobenius algebras are therefore an abstract way to distinguish classical from quantum states by broadcastability.

For this reason, it is useful to have a category of physical systems with special dagger Frobenius algebras to describe the interaction of classical and quantum information. However, special dagger Frobenius algebras are not closed under biproducts, by an argument given in [7]: If A is an n-dimensional matrix algebra, and B is an m-dimensional matrix algebra and A and B are both not normal, the biproduct $A \oplus B$ would have to be normalised by two different numbers at the same time, in order to be isometric. It turns out that in $FHilb$, special dagger Frobenius algebras correspond to normalisable dagger Frobenius algebra, which is a slightly weaker abstract condition that we will explain below. Normalisable dagger Frobenius algebras are in fact closed under biproducts up to isomorphisms.

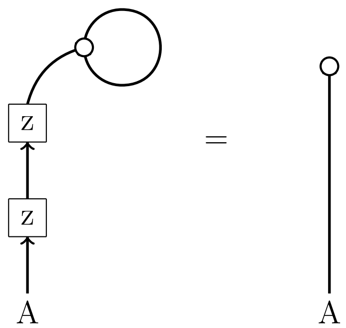
Definition 2.4.5. A map $z : A \to A$ is central for the multiplication $\bigtriangledown$ on A when:



(2.53)

Definition 2.4.6. Normalisable dagger Frobenius algebra

A dagger Frobenius algebra in a dagger compact category **C** is normalisable when it comes with a central, positive definite map $z : A \to A$ such that

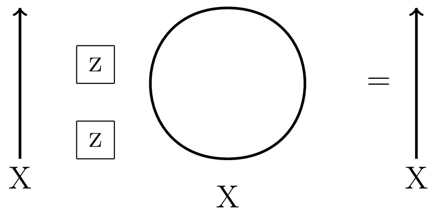


(2.54)

We call a dagger Frobenius algebra normal if $z = \mathrm{id}_A$

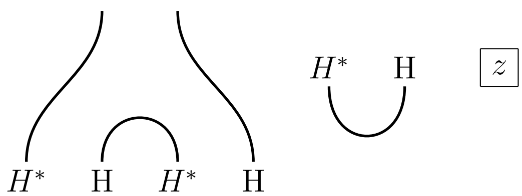
Definition 2.4.7. Positive-dimensional

A category **C** is positive-dimensional, if for every object X of **C**, there is a positive-definite map $z : I \to I$, such that



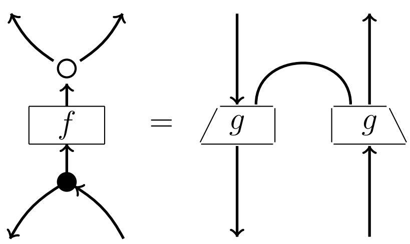
(2.55)

If **C** is a positive dimensional compact category, by proposition 2.11 of [7], every object of the form $H^* \otimes H$ carries a canonical normalisable dagger Frobenius algebra with the following multiplication, unit and normalising morphism:



(2.56)

Definition 2.4.8. Let **C** be a dagger compact category. A completely positive morphisms in **C** between dagger Frobenius algebras $(A, \bigtriangledown_A, u_A)$ and $(B, \bigtriangledown_B, u_B)$ is a morphisms $A \xrightarrow{f} B$ such that for some morphism g,



(2.57)

Here the black dot represents $\bigtriangledown_A$ and the white represents $\bigtriangledown_B$.

Completely positive morphisms between normal dagger Frobenius algebras are called normalised.

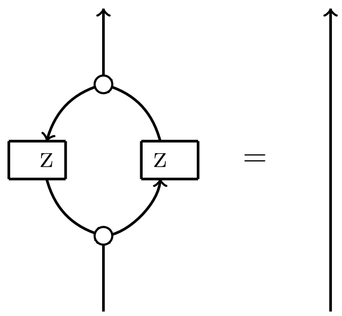
Definition 2.4.9. Let $\mathbf{C}$ be a dagger compact category. The category $CP^*(\mathbf{C})$ is the category of normalisable dagger Frobenius algebras and completely positive maps.

$CP^*_n(\mathbf{C})$ is the subcategory of $CP^*(\mathbf{C})$ that consists of normal dagger Frobenius algebras and normalized completely positive maps.

By [10], every normalisable dagger Frobenius algebra is isomorphic to a normal dagger Frobenius algebra in $CP^*(\mathbf{C})$. It follows that $CP^*(\mathbf{C})$ is dagger isomorphic to $CP^*_n(\mathbf{C})$. If $(A, \bigtriangledown_A, \perp_A)$ and $(B, \bigtriangledown_B, \perp_B)$ are Frobenius algebras, $((A \otimes B), (\bigtriangledown_A \otimes \bigtriangledown_B), (\perp_A \otimes \perp_B))$ is again a Frobenius algebra. The tensor also preserves normalisability.

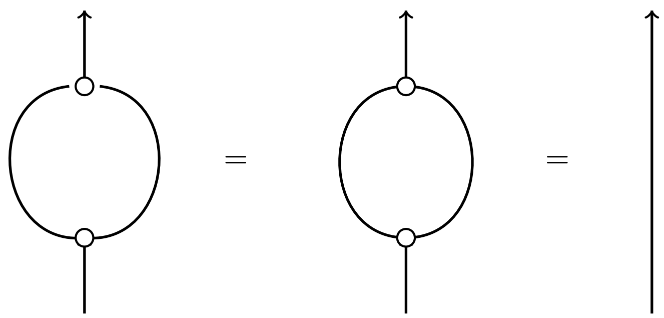
Lemma 2.4.10. *Every normal dagger Frobenius algebra is special*

Proof. By proposition 2.7 of [7], every normalisable dagger Frobenius algebra is symmetric. By lemma 2.10 of [7], for every normalisable dagger Frobenius algebra we have the equality



(2.58)

So for normal dagger Frobenius algebras we have



(2.59)

The first equality is because of the spider rule for symmetric Frobenius algebras [12] and the second by 2.10 of [7] in the special case that z is the identity. □

Algebraically, this category is very elegant, as it unifies the concepts of classical and quantum data by expressing them as independent systems, which are both special cases of the same structure. It is also analogous to a common perspective in physics: to represent quantum systems by operators on these systems. However, it does not have an obvious physical interpretation, especially when we talk about the composition of operators.

2.4.4 Dagger Frobenius algebras in Rel

In this section, we will describe the structure of dagger Frobenius algebras in the category *Rel*. The lemmas regarding groupoids in this section are generally known, however, we are not aware of any written reference. For completeness, we have decided to work out their proofs and include them as lemmas in this dissertation. Commutative dagger Frobenius algebras in *Rel* have been described in [14]. The lemmas in this section regarding general dagger Frobenius algebras in *Rel* are original work.

Special dagger Frobenius algebras $(A, \bigtriangledown, \perp)$ of *Rel* correspond to groupoids in the following sense: A corresponds to the collection of arrows of a groupoid, the unit $I \xrightarrow{\perp} A$ corresponds to the relation $\{(*, u)|u \in U\}$, where U is the collection of unit arrows of A. The multiplication $\bigtriangledown$ is given by composition of the arrows. This is described in [9]. It is straightforward that every dagger Frobenius algebra in *Rel* is normal.

The canonical dagger Frobenius algebra of elements A of $CPM(\mathbf{Rel})$ are indiscrete small groupoids. That is: indiscrete groupoids such that there is only one morphism between every two objects. A is the set of objects of the groupoid. $A \otimes A = \{(a, b)|a, b \in A\}$ is the set of morphisms $a \to b$. The map $A \otimes A \xrightarrow{\bigtriangledown} B^* \otimes B$ is by definition the relation $((a, b, c, d) \sim (a, d)|b = c)$, which is the composition $(c, d) \circ (a, b)$ if and only if $b = c$.

Every groupoid is equivalent to a disjoint union of groups; this is a result of the following lemmas about indiscrete groupoids, which are groupoids such that none of the hom-sets is empty.

Lemma 2.4.11. *All hom-sets of an indiscrete groupoid G are isomorphic*

Proof. We will prove that for any three objects A, B, C of G, $G(A, C) \cong G(B, C)$. Similarly, $G(B, C) \cong G(B, D)$ and therefore $G(A, C) \cong G(B, D)$ for any 4 objects A,B,C,D of G.

G is indiscrete, so we can choose an arrow $A \xrightarrow{f} B$. Since G is a groupoid, f has an inverse $B \xrightarrow{f^{-1}} A$. We can define the isomorphism $F : G(A, C) \to G(B, C)$ as $F(g) = gf^{-1}$, which has the inverse $F^{-1} : G(B, C) \to G(A, C)$ as $F(h) = fh$. □

Groups are categories with one object, where every morphism has an inverse. The elements of the group correspond to the morphisms of the category and multiplication is given by composition of morphisms. Since G is a groupoid, every morphism of G has an inverse, hence the hom-category $G(A, A)$ for any object A of G is a group. The isomorphism constructed above between any two hom-sets of the form $G(A, A)$ and $G(B, B)$ is a group isomorphism, since for $f : A \to B$, and $F : G(A, A) \to G(B, B)$ defined as $F(g) = f \circ g \circ f^{-1}$, if $g, h \in G(A, A)$, then $F(g \cdot h) = f \circ g \circ h \circ f^{-1} = f \circ g \circ f^{-1} \circ f \circ h \circ f^{-1} = F(g) \cdot F(h)$.

This induces a group structure on any hom-set $G(B, C)$ given by $f \cdot g = \xi^{-1}(\xi(f) \cdot \xi(g))$, where $\xi : G(B, C) \to G(A, A)$ is an isomorphism. This isomorphism preserves the group structure by construction; therefore all hom-sets are isomorphic as groups.

Theorem 2.4.12. *Every indiscrete groupoid G is equivalent to the group $G(A, A)$ for any object A of G.*

Proof. As none of the hom-sets are empty, we can choose some morphism $f_B \in G(A, B)$ for every object B of G. Let $F : G \to G(A, A)$ be the functor which is defined on objects $B \in Ob(G)$ as $F(B) = A$ and on morphisms $B \xrightarrow{g} C$ as $F(g) = f_C^{-1} g f_B$. Let $H : G(A, A) \to G$ be the inclusion functor.

$FH \cong \mathrm{id}_{G(A,A)}$ by the natural transformation μ, where $\mu_A = f_A : GH(A) \to \mathrm{id}(A)$. This is a natural isomorphism, as f_A has an inverse.

$HF \cong \mathrm{id}_G$ by the natural transformation ν, where $\nu_B = f_B$. This is a natural isomorphism as f_B is an isomorphism for every object B of G. □

Every groupoid is a disjoint union of indiscrete groupoids; therefore, it is equivalent to a disjoint union of groups.

Corollary 2.4.13. *The canonical Frobenius algebra on any object of $CPM(\mathbf{Rel})$ is equivalent to the unit Frobenius algebra (I, λ_I).*

Proof. The canonical Frobenius algebra C on any object of $CPM(\mathbf{Rel})$ is an indiscrete groupoid. By the previous theorem, it is isomorphic to $C(A, A)$ for any object A of C, which is the group with one object and one morphism, which equals $(I, \lambda_I$ as I is the one element set $\{*\}$ □

Lemma 2.4.14. *For Abelian groupoids, this equivalence is an isomorphism.*

Proof. All arrows in an Abelian groupoid are automorphisms on an object. Suppose there exists an arrow $A \xrightarrow{f} B$. By commutativity, $\mathrm{id}_A \circ f = f \circ \mathrm{id}_A$, but $f \circ \mathrm{id}_A$ is only defined if A is the target of f, so $B = A$. It follows that the groupoid is the disjoint union of the Abelian groups corresponding the objects of the category. □

As classical structures are commutative special dagger Frobenius algebras and thus Abelian groupoids, it follows that classical structures are disjoint unions of Abelian groups. This was observed by [14].

2.5 Functors between monoidal categories

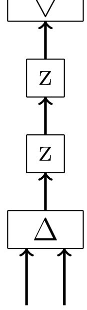
Recall that if $\mathbf{C}$ is a positive dimensional category, every object A of $CPM(\mathbf{C})$ has a canonical Dagger Frobenius algebra (Δ_A, u_A, z_A). In [7] it is shown that the functor $\Gamma : CPM(\mathbf{C}) \to CP^*(\mathbf{C})$ given by $\Gamma(A) = (A, \Delta_A, u_A, z_A)$ and $\Gamma(f) = f$ is full, faithful and dagger symmetric monoidal.

In [10] the authors prove that the functor Γ lifts to a full and faithful functor $\bar{\Gamma} : CP^{\oplus}(\mathbf{C}) \to CP^*(\mathbf{C})$ by proving that $CP^*(\mathbf{C})$ is a biproduct category. The biproduct of dagger Frobenius algebras (A, Δ_A) and (B, Δ_B) is given by $(A \oplus B, \Delta_{A\oplus B}$, where $A \oplus B$ is the biproduct of A and B in $\mathbf{C}$ and $\Delta_{A\oplus B} = \begin{pmatrix} \Delta_A \circ (p_A \otimes p_A) \\ \Delta_B \circ (p_B \otimes p_B) \end{pmatrix} : (A \oplus B) \otimes (A \oplus B) \to (A \oplus B)$ and $\top_{A\oplus B} = \begin{pmatrix} \top_A \\ \top_B \end{pmatrix}$. Now the lift follows from the universal property of the biproduct completion.

In [10] the authors observe that this functor is an equivalence for $FHilb$. $CP^*(FHilb)$ is the category of C^* algebras by [19], CPM($FHilb$) is the category of finite-dimensional matrix algebras and every C^* algebra in $FHilb$ is a direct sum of matrix algebras. However, this is not true in general. By the previous section, dagger Frobenius algebras in *Rel* are groupoids, all dagger Frobenius algebras in *Rel* are normal and the canonical dagger Frobenius algebras on elements of $CPM(Rel)$ are indiscrete groupoids. Not all groupoids can be written as biproducts of indiscrete groupoids.

On the other hand, we have an embedding $\Gamma : CP^*(\mathbf{C}) \to Split^{\dagger}(CPM(\mathbf{C}))$ that is given by $\Gamma(A, \Delta, \perp, z) = (A, P_{A,\Delta,\perp,z})$ on objects, where

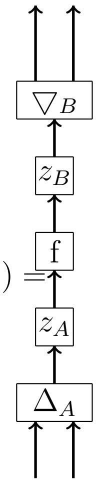
$P_{A,\Delta,\perp,z} =$



(2.60)

and

$$\Gamma((A, \Delta_A, \perp_A, z_A) \xrightarrow{f} (B, \Delta_B, \perp_B, z_B)) =$$



(2.61)

By [10] this embedding is full, faithful and strongly dagger symmetric monoidal. It is still an open question whether or not this embedding is an equivalence for $FHilb$. However, this is likely to be the case because otherwise there exist other structures than classical, quantum, or mixtures of these.

Chapter 3

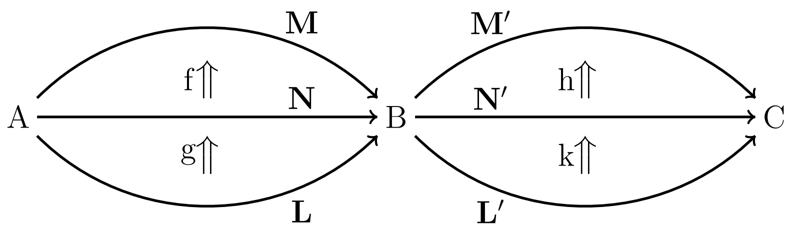
2-categorical models

In this chapter, we will develop a module-theoretic construction of a weak 2-category, from any monoidal dagger category $\mathbf{C}$. We will call this 2-category $2(\mathbf{C})$. The category $2(\mathbf{C})$ can serve as a model for quantum theory, alternative to the monoidal categories discussed in chapter 2. The motivation for this abstract categorical model, is that physical systems can be described as pure quantum systems that interact with their environment. In [20] Vicary formalises these principles abstractly, which leads to a description of quantum systems in terms of dagger bimodules of classical structures. The resulting mathematical model is a 2-category, in which physical systems are described by modules on classical structures, and physical operations as module homomorphisms. The category $2(\mathbf{C})$ gives us a detailed mathematical framework for the ideas developed in [20]. It is slightly more general, as we will define physical systems as modules on dagger Frobenius algebras, instead of classical structures. If we restrict to classical structures, this construction gives us a formal definition of the category described in [20]. If we do not restrict to classical structures, this gives an extra layer to our model. In that case, we allow the environment to exhibit quantum behaviour as well. This is not in correspondence with experiments, but it corresponds to the idea that the environment is another physical system, which is not fundamentally different from any quantum system. This generalisation turns out to be useful to expose the relation between the 2-categorical framework and the monoidal category $CP^*(\mathbf{C})$. When we apply the construction $2(-)$ to the category $FHilb$, we obtain a category in which $2Hilb$ can be embedded up to equivalence.

All 2-categories in this chapter are weak 2-categories. Sometimes we will just call them 2-categories, for readability.

3.1 2-categories

Definition 3.1.1. A weak 2-category is a structure that has 0-cells: $A, B, C, ...$, 1-cells:$\mathbf{M}, \mathbf{N}, ...$, and 2-cells: $f, g, h, ...$. The 1-cells are morphisms between the 0-cells and the 2-cells are morphisms between 1-cells from the same hom-set. We can compose 1-cells 'horizontally' and 2-cells both 'horizontally' and 'vertically'. This is illustrated in the following picture:



(3.1)

We write $A \xrightarrow{N} B \xrightarrow{N'} C$ or $N \otimes_B N'$ and $f \otimes_B h$ for the horizontal composition of 1-cells and 2-cells, respectively. We write $f \circ g$ for the vertical composition of 2-cells. This structure satisfies the following conditions:

1 1-cells are closed under horizontal composition and 2-cells are closed under both horizontal and vertical composition.

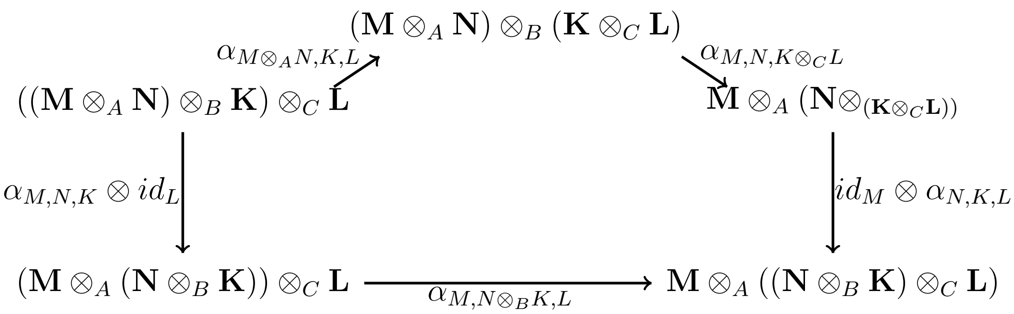
2 There is an invertible 2-cell $\alpha_{M,M',M''}$:$(\mathbf{M} \otimes_B \mathbf{M}') \otimes_C \mathbf{M}'' \Rightarrow \mathbf{M} \otimes_B (\mathbf{M}' \otimes_C \mathbf{M}'')$ for every triple of 1-cells $A \xrightarrow{\mathbf{M}} B \xrightarrow{\mathbf{M}'} C \xrightarrow{\mathbf{M}''} D$. This 2-cell is natural with respect to its arguments.

3 For every 0-cell $\mathbf{A}$ there is a 1-cell $\mathbf{A}$ such that for every 1-cell $A \xrightarrow{\mathbf{M}} B$ there are two invertible 2-cells $l_M : \mathbf{A} \otimes_A \mathbf{M} \Rightarrow \mathbf{M}$ and $r_M : \mathbf{M} \otimes_B \mathbf{B} \Rightarrow \mathbf{M}$. These morphisms are natural with respects to their arguments.

4 Vertical composition is associative and for every 1-cell $\mathbf{M}$ there is a 2-cell id_M that is the identity for vertical composition.

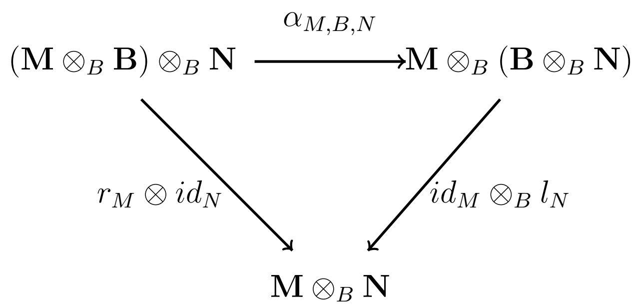
5 Vertical and horizontal composition satisfy the exchange law.

Furthermore, the 2-cells of a weak 2-category satisfy the pentagon equality and the triangle equality. The pentagon equality is given by the following diagram:



(3.2)

The triangle equality is the diagram:



(3.3)

We can regard a monoidal category as a 2-category with one 0-cell. Objects of the monoidal category are the 1-cells, and the morphisms are the 2-cells. Horizontal composition is given by the tensor product, and vertical composition corresponds to the composition of morphisms in the monoidal category. Throughout this chapter we will discuss relations between certain 2-categories, and certain monoidal categories. For this purpose, we will regard all monoidal categories as 2-categories. By doing this, we make it possible to talk about functors between 2-categories and monoidal categories. Functors from and to weak 2-categories need only to be defined up to isomorphisms of 0-cells and 1-cells, as 0-cells and 1-cells in weak 2-categories are only defined up to isomorphisms.

3.2 The construction $2(-)$

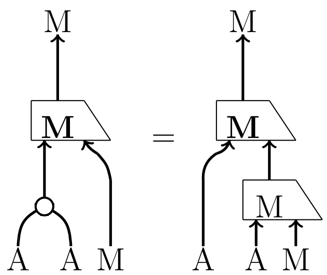
In this section, we will give the definition of a 2-category that can serve as a model for quantum theory. This is a construction that turns any monoidal dagger category $\mathbf{C}$ in the 2-category $2(\mathbf{C})$. In order to do this, we first need to define several concepts concerning modules. Finally we will look at the structure of modules in the special case that $\mathbf{C}$ has biproducts.

3.2.1 Dagger Modules

In our formalism, we will describe physical systems as pure quantum systems, that interact with their environment. If through this interaction information about the quantum system is transferred to its environment, then the quantum system behaves classically with respect to this environment. We model the environment as a separate physical system, which is described by a dagger Frobenius algebra. The interaction of the pure system with its environment obeys certain principles. As we would expect, if the quantum system interacts with its environment twice, the second time will not change its state. This is because after the first time, the system already behaves classically with respect to its environment. Furthermore, if somehow the information about the system that is transferred to the environment is erased, then the quantum system is again in its original superposition of states. This corresponds to the outcome of experiments concerning quantum erasure, one such experiment is described in [21].

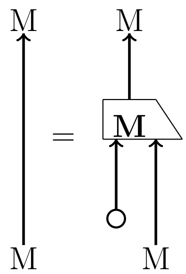
Modules are a precise way of defining this interaction of a quantum system with its environment.

Definition 3.2.1. Let **C** be a monoidal dagger category. A left module M of a Frobenius algebra A on an object X is a morphism $\mathbf{M} : A \otimes M \to M$ in **C** that satisfies the following equations:



(3.4)

,



(3.5)

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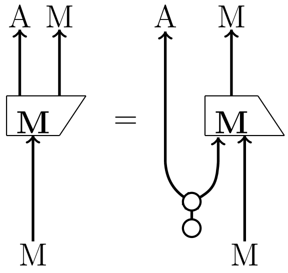
A left module is called a left dagger module if



(3.6)

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A right module $\mathbf{M}$ is a morphism $\mathbf{M} : M \otimes A \to M$ that satisfies similar equations.

There is no fundamental difference between left and right modules; therefore we will often just talk about modules.

Observe that by these conditions, a dagger module is exactly the dagger of a measurement. This is what we would expect, as we interpret a measurement as an interaction with the environment.

Equation 3.4 formalises the condition that a second interaction of the system with its environment will not change the system. Equation 3.5 and 3.6 correspond to the condition that if the transferred information is somehow deleted, this recovers the original state of the quantum system.

In our model, we allow interaction of quantum systems with two different environments. We assume that when the system interacts with these two separate environments, on after the other, the order in which this happens does not matter.

In order to describe the interaction of a quantum system with more than one other system mathematically, we extend the definition of a module to a bimodule.

Definition 3.2.2. Let $\mathbf{C}$ be a symmetric monoidal dagger category. A bimodule of Frobenius algebras A, B is a morphism $\mathbf{M} : A \otimes M \otimes B \to M$, where $\mathbf{M} = \mathbf{M'} \circ \mathrm{id}_A \otimes \sigma_{M,B}$ and $\mathbf{M'}$ is a left module of $A \otimes B$ on M.

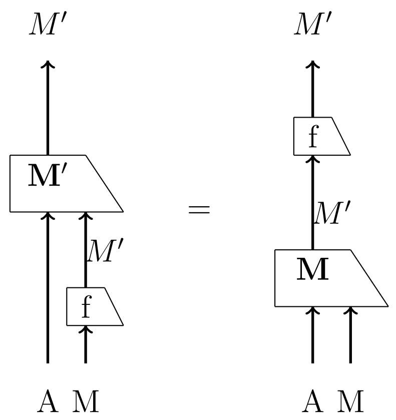
$\mathbf{M}$ is a dagger bimodule, if $\mathbf{M'}$ is a left dagger module.

This definition implicitly captures the condition that the two interactions are commutative, as they are described simultaneously.

Any (dagger) bimodule $A \otimes M \otimes B \xrightarrow{\mathbf{M}} M$ is naturally isomorphic to the (dagger) module $(A \otimes B) \otimes M \xrightarrow{\mathbf{M'}} M$, by the natural isomorphism $\mathrm{id}_A \otimes \sigma_{B,M}$. Hence, whenever we talk about dagger modules, this is just as general as when we would talk about dagger bimodules and the other way around. For simplicity, we will restrict to dagger modules whenever that is possible.

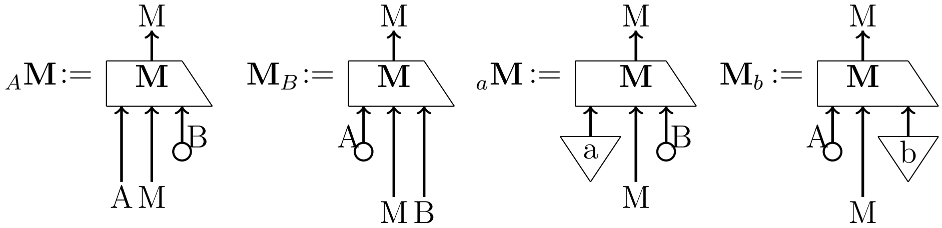
In our formalism we require morphisms between pure quantum systems to commute with the interaction of the system with its environment. This means that if we apply an operation before an interaction of the system with the environment, we get the same outcome as when we apply the operation after the system interacts with the environment. These operations are said to be protected from decoherence. In quantum computing, we need this type of operation to write a well-defined quantum protocol because we cannot control whether the operation happens before or after an interaction with the environment. In other words, operations that are protected from decoherence correspond to deterministic operations. The following definition makes this precise.

Definition 3.2.3. Let $\mathbf{C}$ be a monoidal dagger category. A module homomorphism $f : \mathbf{M} \to \mathbf{M}'$ between the modules $\mathbf{M} : A \otimes M \to M$ and $\mathbf{M}' : A' \otimes M' \to M'$ is a morphism $f : M \to M'$ in $\mathbf{C}$ on the underlying objects M, M' of $\mathbf{M}, \mathbf{M}'$, respectively, such that the following equality holds:



(3.7)

We will always write modules $\mathbf{M}$ in bold letters and their underlying object M in normal letters. Furthermore, we will make use of the following notation:



(3.8)

Moreover, we will write $M_a := \mathbf{M}_a(M)$

The bimodules of a symmetric monoidal dagger category $\mathbf{C}$ will play an important role in the 2-category $2(\mathbf{C})$. The notation $\mathbf{M}$, $_A\mathbf{M}$, $\mathbf{M}_B$, $_a\mathbf{M}$ and $\mathbf{M}_b$ will be used to refer to both the module in $2(\mathbf{C})$ and to its corresponding morphism in $\mathbf{C}$. Similarly, we will use the same notation for module homomorphisms $\mathbf{M} \xrightarrow{f} \mathbf{M'}$ and the corresponding morphism $M \xrightarrow{f} M'$ in $\mathbb{C}$ between the underlying objects M, M'. Whenever this is not clear from the context, we will specify which concept we are referring to.

3.2.2 Tensor product of modules

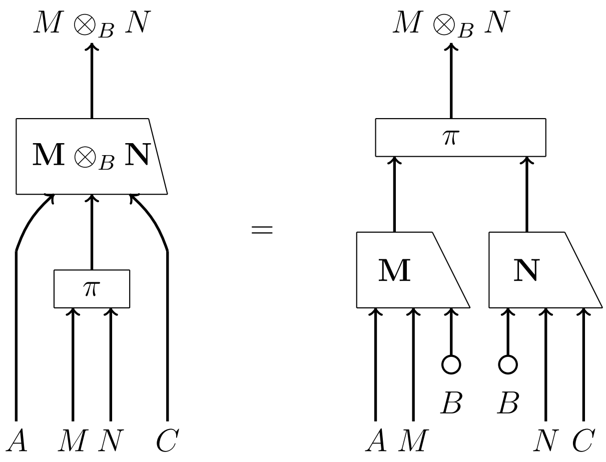
If one of two entangled quantum systems interacts with its environment, then this determines the possible joint states of the compound system. One of the characteristics of entanglement, is that a measurement on any of the two states determines the other state. Hence, it should not make any difference which of the two entangled systems interacts with the environment.

In our formalism, we describe the joint system with respect to the environment, as the biggest subspace in which the interaction of one state with the environment has the same result as the interaction of the other state with the environment. We call this the tensor product with respect to the environment. Mathematically this is described by the coequaliser of the two interactions. To ensure the existence of this tensor product, we require the coequaliser to be a dagger coequaliser. This is a coequaliser $A \xrightarrow{\pi} B$ in a dagger category, such that $\pi^\dagger \circ \pi = \mathrm{id}_A$.

Definition 3.2.4. Let $\mathbf{C}$ be a dagger category where all coequalisers exist, and where all coequalisers are dagger coequalisers. Let $\mathbf{M}$ be a bimodule of A and B on M, let $\mathbf{N}$ be a bimodule of B and C on N, and let π be the coequaliser given by the following picture

$$\begin{array}{ccccc} M \otimes B \otimes N & \overset{\mathbf{M}_B \otimes id_N}{\underset{id_M \otimes\, {}_B\mathbf{N}}{\rightrightarrows}} & M \otimes N & \xrightarrow{\pi} & M \otimes_B N \\ & & & \searrow^{\mathrm{f}} & \downarrow{\scriptstyle \tilde{f}} \\ & & & & \mathrm{K} \end{array} \tag{3.9}$$

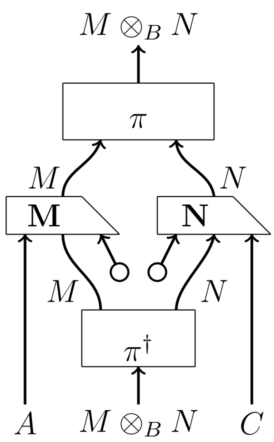
The tensor product of $\mathbf{M}$ and $\mathbf{N}$ with respect to B is defined as the unique module $\mathbf{M} \otimes_B \mathbf{N}$, of A and C on $M \otimes_B N$, such that the corresponding morphism in $\mathbf{C}$ satisfies the following equality:



(3.10)

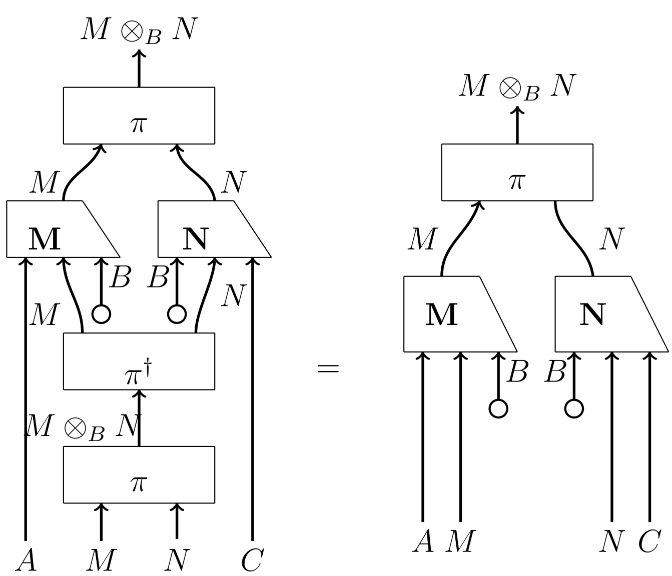
We need to check that the tensor product $\mathbf{M} \otimes_B \mathbf{N}$ exists, and that it is well-defined.

First of all, the module



(3.11)

satisfies equation 3.10 because



(3.12)

as $\pi^\dagger \circ \pi = \mathrm{id}_{M \otimes N}$. Observe that this is indeed a module; all module axioms hold because ${}_A\mathbf{M} \otimes \mathbf{N}_C$ is a module, and because $\pi^\dagger \circ \pi = \mathrm{id}_{M \otimes N}$.

This proves that $\mathbf{M} \otimes_A \mathbf{N}$ exists.

Theorem I of section 3.10 of [8] states that every equaliser is monic. Coequalisers are the duals of equalisers, so every coequaliser is an epic morphism. This means that if there are two morphisms $\mathbf{M} \otimes_B \mathbf{N}$, and $(\mathbf{M} \otimes_B \mathbf{N})'$ in $\mathbf{C}$ that satisfy the equality above, then $\mathbf{M} \otimes_B \mathbf{N} = (\mathbf{M} \otimes_B \mathbf{N})'$. As a consequence, π is a module homomorphism.

We can extend this to tensor products of morphisms.

Definition 3.2.5. Let $\mathbf{C}$ be a symmetric monoidal dagger category, where all coequalisers exist, and where all coequalisers are dagger coequalisers. Let $\mathbf{M}$ be a bimodule of A and B on M, let $\mathbf{N}$ be a bimodule of B and C on N. Let $f : M \to M'$ and $g : N \to N'$ be morphisms in $\mathbf{C}$ that correspond to module homomorphisms $\mathbf{M} \xrightarrow{f} \mathbf{M}'$, $\mathbf{N} \xrightarrow{g} \mathbf{N}'$. Let $\pi : M \otimes N \to M \otimes_B N$ $\phi : M' \otimes N' \to M' \otimes_B N'$ be the coequalisers that define the tensor products $\mathbf{M} \otimes_{\mathbf{B}} \mathbf{N}$ and $\mathbf{M}' \otimes_{\mathbf{B}} \mathbf{N}'$ respectively. These morphisms form the diagram below:

$$\begin{array}{ccccc} M \otimes B \otimes N & \underset{id_M \otimes\ _B\mathbf{N}}{\overset{\mathbf{M}_B \otimes id_N}{\rightrightarrows}} & M \otimes N & \xrightarrow{\pi} & M \otimes_B N \\ \big\downarrow{\scriptstyle f \otimes id_B \otimes g} & & \big\downarrow{\scriptstyle f \otimes g} & & \big\downarrow{\scriptstyle f \otimes_B g} \\ M' \otimes B \otimes N' & \underset{id'_M \otimes\ _B\mathbf{N}'}{\overset{\mathbf{M}'_B \otimes id_{N'}}{\rightrightarrows}} & M' \otimes N' & \xrightarrow{\phi} & M' \otimes_B N' \end{array} \tag{3.13}$$

The left square of the diagram commutes because f and g are module homomorphisms. The morphism ϕ is the coequaliser of $\mathbf{M}'_B \otimes \mathrm{id}_{N'}$ and $\mathrm{id}'_M \otimes\ _B\mathbf{N}'$, so $\phi \circ (\mathbf{M}'_B \otimes \mathrm{id}_{N'}) \circ (f \otimes \mathrm{id}_B \otimes g) = \phi \circ (\mathrm{id}'_M \otimes\ _B\mathbf{N}') \circ (f \otimes \mathrm{id}_B \otimes g)$. By commutativity of the left sub-diagram, $\phi \circ (f \otimes g) \circ (\mathbf{M}_B \otimes \mathrm{id}_N) = \phi \circ (f \otimes g) \circ (\mathrm{id}_M \otimes\ _B\mathbf{N})$. Since π is a coequaliser, there exists a unique morphism that makes the sub-diagram on the right commute. We define the tensor product $f \otimes_B g$ to be this unique morphism. This is a module homomorphism because π, ϕ and $f \otimes g$ are module homomorphisms.

Lemma 3.2.6. *Let $\mathbf{C}$ be a dagger category where all coequalisers exist, and where all coequalisers are dagger coequalisers. Let $\mathbf{M}$ be a bimodule of A and B on M, let $\mathbf{N}$ be a bimodule of B and C on N. We have the equality* $\mathrm{id}_M \otimes_B \mathrm{id}_N = \mathrm{id}_{\mathbf{M} \otimes_B \mathbf{N}}$.

Proof. The equality holds because both morphisms make the diagram below commute:

$$
\begin{array}{ccccc}
M \otimes A \otimes N & \overset{\mathbf{M}_A \otimes id_N}{\underset{id_M \otimes {}_A\mathbf{N}}{\rightrightarrows}} & M \otimes N & \xrightarrow{\pi} & M \otimes_B N \\
\Big\downarrow {\scriptstyle \mathrm{id}_M \otimes \mathrm{id}_A \otimes \mathrm{id}_N} & & \Big\downarrow {\scriptstyle \mathrm{id}_M \otimes \mathrm{id}_N} & & \vdots \\
M \otimes A \otimes N & \overset{\mathbf{M}_A \otimes id_N}{\underset{id_M \otimes {}_A\mathbf{N}}{\rightrightarrows}} & M \otimes N & \xrightarrow{\pi} & M' \otimes_B N'
\end{array}
\tag{3.14}
$$

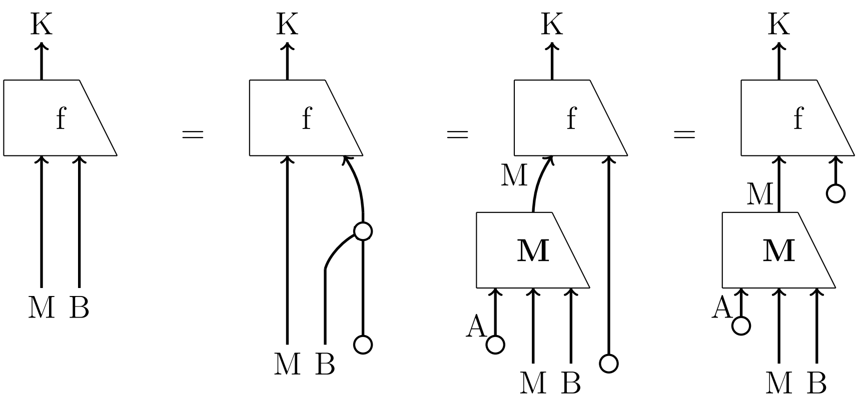
□

Lemma 3.2.7. *Let* $\mathbf{C}$ *be a symmetric monoidal dagger category, where all coequalisers exist, and where all coequalisers are dagger coequalisers. Let* $\mathbf{A} : I \otimes A \otimes A \to A$, $\mathbf{M} : A \otimes M \otimes B \to M$, *and* $\mathbf{B} : B \otimes B \otimes I \to B$ *be dagger modules of the dagger Frobenius algebras* $(A, \Delta'), (B, \Delta)$, *and* (I, λ_I), *where* $\mathbf{A}_A = \Delta'$ *and* ${}_B\mathbf{B} = \Delta$. *Let* π_A *be the coequaliser of* $\mathbf{A}_A \otimes \mathrm{id}_M$ *and* $\mathrm{id}_A \otimes {}_A\mathbf{M}$. *Let* π_B *be the coequaliser of* $\mathbf{M}_B \otimes \mathrm{id}_B$ *and* $\mathrm{id}_B \otimes {}_B\mathbf{B}$. *There exist canonical unitary 2-cells* $\mathbf{M} \otimes_B \mathbf{B} \xrightarrow{\beta} {}_A\mathbf{M}$ *and* $\mathbf{A} \otimes_A \mathbf{M} \xrightarrow{\gamma} \mathbf{M}_B$, *such that* $\beta \circ \pi_B = \mathbf{M}_B$ *and* $\beta^\dagger \circ \mathbf{M}_B = \pi_B$, *and* $\gamma \circ \pi_A = {}_A\mathbf{M}$ *and* $\gamma^\dagger \circ {}_A\mathbf{M} = \pi_A$.

Proof. We will show that $\mathbf{M}_B$ is a coequaliser of $\mathbf{M}_B \otimes \mathrm{id}_B$, and $\mathrm{id}_B \otimes {}_B\mathbf{B}$. Furthermore, we show that ${}_A\mathbf{M}$ is the unique morphism such that equation 3.10 holds. We will prove that the canonical unitary 2-cell β then follows from the uniqueness of coequalisers. The proof of the existence of γ is very similar, so we will leave this out.

Observe that $\mathbf{M}_B \circ (\mathbf{M}_B \otimes \mathrm{id}_M) = \mathbf{M}_B \circ (\mathrm{id}_M \otimes \Delta) = \mathbf{M}_B \circ (\mathrm{id}_B \otimes {}_B\mathbf{B})$. The first equality is by equation 3.4, and the second by definition of ${}_B\mathbf{B}$.

It is left to show that whenever we have a map $f : M \otimes B \to K$, such that $f \circ (\mathbf{M}_B \otimes \mathrm{id}_B) = f \circ (\mathrm{id}_M \otimes {}_B\mathbf{B})$, it factors through $\mathbf{M}_B$. We have the equalities

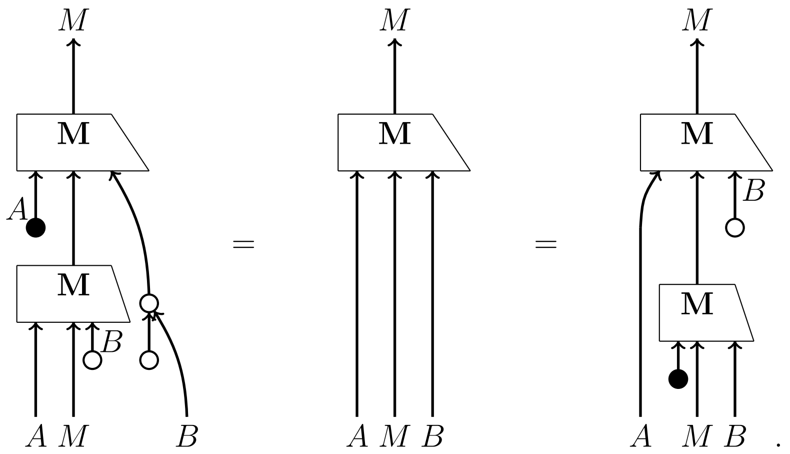


(3.15)

The second equality follows from our assumption that $f \circ (\mathrm{id}_M \otimes {}_B\mathbf{B}) = f \circ (\mathbf{M}_B \otimes \mathrm{id}_B)$.

So $f = f \circ (\mathrm{id}_M \otimes u_B) \circ (\mathbf{M}_B \otimes \mathrm{id}_M)$, and hence f factors through $\mathbf{M}_B$.

The module ${}_A\mathbf{M}$ is the unique morphism such that equation 3.10 holds. This is shown in the following diagram:

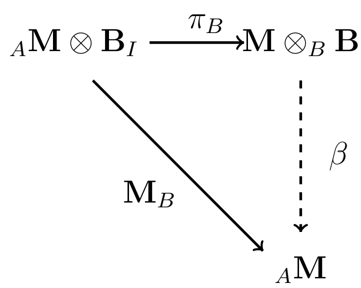


(3.16)

In other words, $\mathbf{M}_B$ is a module homomorphism from ${}_A\mathbf{M} \otimes \mathbf{B}_I$ to ${}_A\mathbf{M}$.

By uniqueness of coequalisers, there exists a unique morphism $M \otimes_B N \xrightarrow{\beta} M$, such that $\beta \circ \pi_B = \mathbf{M}_B$ and $\beta^\dagger \circ \mathbf{M}_B = \pi_B$.

We claim that β is a unitary 2-cell. In that case, we have the commuting diagram below.



(3.17)

The morphism β is unitary, since $\beta^\dagger \circ \beta \circ \pi_B = \mathrm{id}_{M \otimes_B B} \circ \pi_B$. It follows that $\beta^\dagger \circ \beta = \mathrm{id}_{M \otimes_B B}$ because coequalisers are epic. Similarly, $\beta \circ \beta^\dagger = \mathrm{id}_M$.

We use the fact that π_B and $\beta \circ \pi_B = \mathbf{M}_B$ are module homomorphisms to prove that β is a module homomorphism: $\beta \circ (\mathbf{M} \otimes_B \mathbf{B}) \circ (\mathrm{id}_A \otimes \pi_B \otimes \mathrm{id}_N) = \beta \circ \pi_B \circ ({}_A\mathbf{M} \otimes \mathbf{B}_B) = {}_A\mathbf{M} \circ \beta \circ \pi_B$. Now it follows that $\beta \circ (\mathbf{M} \otimes_B \mathbf{B}) =_A \mathbf{M} \circ \beta$ because π_B is epic. Similarly we can prove that $\beta^\dagger$ is a module homomorphism. This proves that β defines a unitary 2-cell. □

3.2.3 2(C)

Now we have developed all the tools we need to define the category $2(\mathbf{C})$. As we will prove later in this section, this is a weak 2-category constructed from an arbitrary monoidal category **C**.

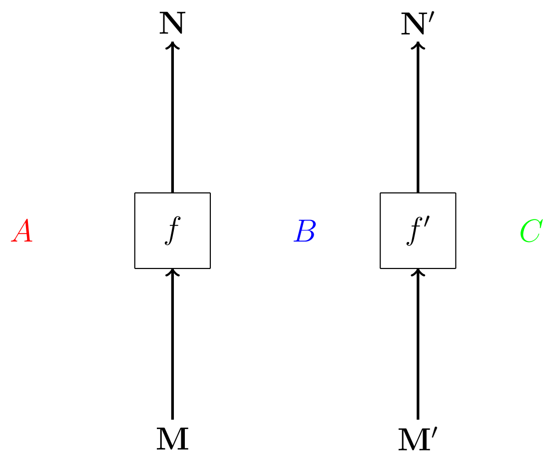
Definition 3.2.8. Let **C** be a symmetric monoidal dagger category, where all coequalisers exist, and where all coequalisers are dagger coequalisers. The category $2(\mathbf{C})$ is the 2-category that consists of the following data: the 0-cells are the dagger Frobenius algebras of **C**; the 1-cells are dagger bimodules of dagger Frobenius algebras of **C** on objects of **C**; and the 2-cells are bimodule homomorphisms between modules in the same hom-sets of 0-cells. Horizontal composition for modules and module homomorphisms is given by the tensor product as defined in definition 3.2.4, and definition 3.2.5. Vertical composition is given by the normal composition of module homomorphisms.

For some theorems, it will be necessary to restrict to 0-cells, which are special dagger Frobenius algebras, normalisable dagger Frobenius algebras or classical structures. In these cases, we will write $2(\mathbf{C})_s$, $2(\mathbf{C})_n$, or $2(\mathbf{C})_c$, respectively.

Most of the time we will write $A, B, C, ...$ for 0-cells. In that case we refer to the objects $A, B, C, .. \in \mathbf{C}$, together with a Frobenius algebra on each object. Only when we need to specify the multiplication of the Frobenius algebra, we will write $(A, \bigtriangledown_A), (B, \bigtriangledown_B), (C, \bigtriangledown_C), ...$

Weak n-categories have a natural diagrammatic language. Diagram 3.1, which describes the coherence of vertical and horizontal composition, can be seen as a simplicial complex, where the 2-cells are the areas between the coherent 1-cells. This holds for any coherent diagram of cells in our 2-category. By the planar Poincaré duality, we can also represent these 0-,1- and 2-cells by their dual block representation. This is a transformation in algebraic topology on simplicial complexes, which turns all vertices into areas and all areas into vertices. We obtain diagrams that are very similar to the ones that were described in chapter 2.

These diagrams are of the following form:



(3.18)

The areas marked with the coloured letters A, B, and C are the 0-cells, so these represent the environment. The lines marked with $\mathbf{M}$, $\mathbf{M}'$, $\mathbf{N}$, and $\mathbf{N}'$ are the 1-cells. The lines $\mathbf{M}$ and $\mathbf{M}'$ represent bimodules of A and B, and $\mathbf{N}$ and $\mathbf{N}'$ represent bimodules of B and C. The boxes f, f' are 2-cells, which represent the module homomorphism from $\mathbf{M}$ to $\mathbf{N}$, and from $\mathbf{M}'$ to $\mathbf{N}'$, respectively. We will only represent 2-cells with a box when we want to specify which 2-cell it is. If this is clear from the context or unimportant, we just write a dot, or we write only a transition between wires and areas. This normally does not cause any ambiguity. In all diagrams, time flows upwards. A vertical composition of 2-cells can be interpreted as subsequent operations, and a horizontal composition of 1-, and 2-cells can be interpreted as a spatial separation between the cells. The diagram represents the horizontal composition of morphisms $f \otimes_B f'$. This is the morphism between the horizontal compositions of modules $\mathbf{M} \otimes_B \mathbf{M}'$ and $\mathbf{N} \otimes_B \mathbf{N}'$.

If we regard monoidal categories as 2-categories with one object, there is only one 0-cell, so we can omit the labels of the 0-cells in the diagrams. The corresponding diagrammatic language, is the language we discussed in chapter 2 for monoidal categories.

Lemma 3.2.9. *Let* $\mathbf{C}$ *be a symmetric monoidal dagger category, in which all coequalisers exist. We write* I *for the unit element* I *with a dagger Frobenius.* $2(\mathbf{C})(I, I)$ *is isomorphic to* $\mathbf{C}$*.*

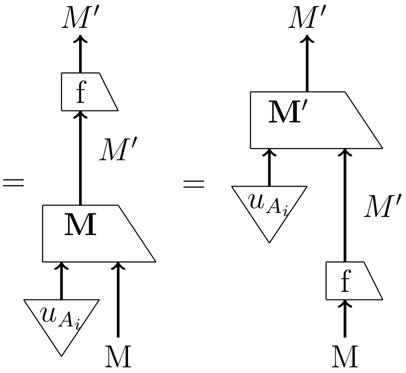
Proof. Let $\mathbf{M}$ be a module on the unit object. This is a scalar multiple of the identity on M. In fact, by 3.5 it is a multiplication by a unitary scalar. It follows that any two modules $\mathbf{M}$, $\mathbf{M}'$, of the unit object on an object M, are isomorphic. Furthermore, every morphism $M \to N$ in $\mathbf{C}$ is a module homomorphism $\mathbf{M} \to \mathbf{N}$ in $2(\mathbf{C})$. Consequently, the forgetful functor $2(\mathbf{C})(I, I) \to \mathbf{C}$ is surjective and injective on 1-cells, up to isomorphism, and also full and faithful on hom-sets of 1-cells. □

3.2.4 Modules in dagger biproduct categories

In this section, we assume that $\mathbf{C}$ is a symmetric monoidal dagger category with biproducts, where all coequalisers exist, and where all coequalisers are dagger coequalisers. Recall that the biproduct $A = \oplus_i A_i$, of Frobenius algebras A_i, with units u_{A_i}, is a Frobenius algebra, with unit $u_A = \sum_i u_{A_i}$. By 3.6, for any module $\mathbf{M}$ of A on M in $\mathbf{C}$, $\mathbf{M} = \sum_i \mathbf{M}_{u_{A_i}}$ and $M = \oplus_i M_{a_i}$. When $\mathbf{M}$ is a bimodule of $A = \oplus_i A_i$ and $B = \oplus_j B_j$, we can write $\mathbf{M}$ as the matrix with coefficients $\mathbf{M}_{i,j} := {}_{a_i}\mathbf{M}_{b_j}$, and its underlying space M, as the matrix with coefficients $M_{i,j} := \mathbf{M}_{i,j}(M)$.

For a module homomorphism $\mathbf{M} \xrightarrow{f} \mathbf{M}'$ in $\mathbf{C}$, we have

$$f \circ \mathbf{M}_{u_{A_i}} = \quad (3.19)$$



$$= \mathbf{M'}_{u_{A_i}} \circ f \qquad (3.19)$$

,

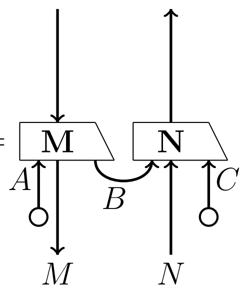
where the second equality is by definition 3.7 about module homomorphisms.

This means we can write f as the sum of the morphisms $\sum_i \mathbf{M}_{u_{A_i}} \xrightarrow{f_i} \mathbf{M'}_{u_{A_i}}$, where $f_i := f \circ \mathbf{M}_{u_{A_i}}$. If $\mathbf{M}$ is a bimodule, we can write f as the matrix with coefficients $\mathbf{M}_{i,j} \xrightarrow{f_{i,j}} \mathbf{M'}_{i,j}$.

Lemma 3.2.10. *Let $\mathbf{C}$ be a compact category with biproducts. Let A, B, C be the biproducts of dagger Frobenius algebras $\oplus_i A_i$, $\oplus_j B_j$, and $\oplus_k C_k$, with units $\sum_i a_i$, $\sum_j b_j$, and $\sum_k c_k$, respectively, such that all b_j are copyable states with respect to the Frobenius algebra B_j. The tensor product $\mathbf{M} \otimes_B \mathbf{N}$, of $\mathbf{M} \in 2(\mathbf{C})(A, B)$ and $\mathbf{N} \in 2(\mathbf{C})(B, C)$, is isomorphic to the module obtained by generalized matrix multiplication $\sum_{b_j \in B} \mathbf{M}_{b_j} \otimes {}_{b_j}\mathbf{N}$.*

Proof. Let $\mathbf{M_B N}$ be the morphism

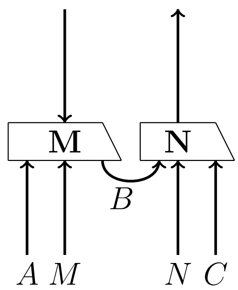
$$\mathbf{M_B N} := \qquad (3.20)$$



,

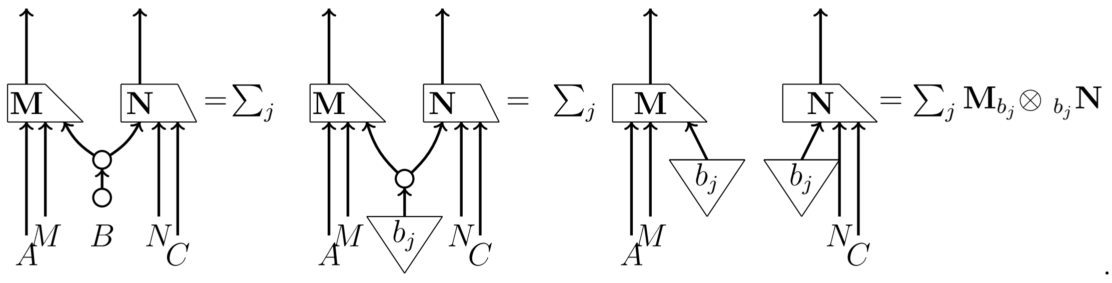
and let ${}_A\mathbf{M}_B\mathbf{N}_C$ be the dagger module defined as

$$(3.21)$$



.

By the diagram below, $\sum_{b_j \in B} \mathbf{M}_{b_j} \otimes {}_{b_j}\mathbf{N}$ equals the module ${}_A\mathbf{M}_B\mathbf{N}_C$.

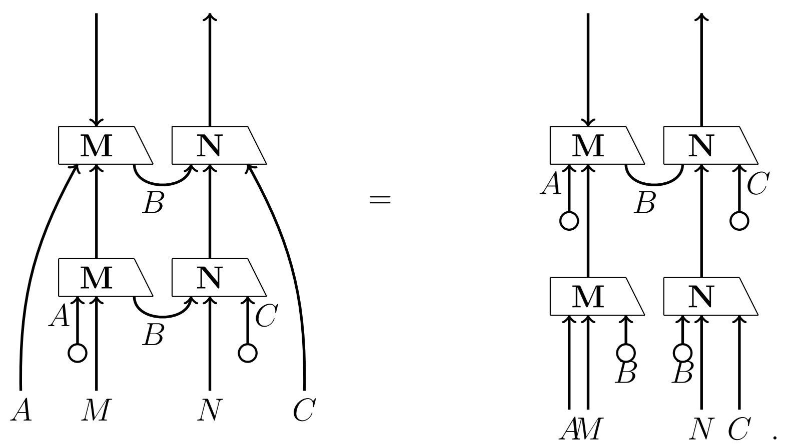


(3.22)

The first equality holds, by the definition of the unit of a biproduct of dagger Frobenius algebras. The second equality holds because all b_j are copyable states.

As a consequence, $\mathbf{M_B N}$ equals $\sum_{a_i, c_k} \sum_{b_j} \mathbf{M}_{i,j} \otimes \mathbf{N}_{j,k}$. Its image is therefore $\oplus_{i,j,k}$ $M_{i,j} \otimes N_{j,k}$.

The module ${}_A\mathbf{M}_B\mathbf{N}_C$ on $\oplus_{i,j,k}$ $M_{i,j} \otimes N_{j,k}$ is the unique morphism that satisfies equation 3.10, which is in this case:



(3.23)

The equality holds by definition 3.4, the Frobenius laws and because b_j are copyable points.

Observe that ${}_A\mathbf{M}_B\mathbf{N}_C$ is in fact a module. It satisfies 3.4 because $\mathbf{M}$ and $\mathbf{N}$ are modules and because b_j are copyable points; it satisfies 3.5 because $\mathbf{M}$ and $\mathbf{N}$ are modules and $\mathbf{M_B N}$ equals $\sum_{a_i, c_k} \sum_{b_j} \mathbf{M}_{i,j} \otimes \mathbf{N}_{j,k}$; it satisfies 3.6 because $\mathbf{M}$ and $\mathbf{N}$ are dagger modules.

There exists a canonical module homomorphism $\mathbf{M} \otimes_B \mathbf{N} \xrightarrow{\gamma_{\mathbf{M},\mathbf{N}}} {}_A\mathbf{M}_B\mathbf{N}_C$, such that $\gamma_{\mathbf{M},\mathbf{N}} \circ \pi = \mathbf{M_B N}$ and $\gamma^{\dagger}_{\mathbf{M},\mathbf{N}} \circ \mathbf{M_B N} = \pi$.

We have the following commuting diagram:

$$\begin{array}{ccc} \mathbf{M}\otimes\mathbf{N} & \xrightarrow{\pi} & \mathbf{M}\otimes_B\mathbf{N} \\ & \searrow^{\mathbf{M_B N}} & \Big\downarrow \gamma \\ & & \sum_{i,j,k}\mathbf{M}_{i,j}\otimes\mathbf{N}_{j,k}, \end{array} \tag{3.24}$$

The morphisms γ and $\gamma^\dagger$ are module homomorphisms because π and $\mathbf{M_B N}$ are epic module homomorphisms. Furthermore, we have that $\gamma^\dagger \circ \gamma \circ \pi = \pi = \mathrm{id}_{\mathbf{M}\otimes_B\mathbf{N}} \circ \pi$. Because coequalisers are epic, $\gamma^\dagger \circ \gamma = \mathrm{id}_{\mathbf{M}\otimes_B\mathbf{N}}$. Similarly, $\gamma \circ \gamma^\dagger = \mathrm{id}_{\sum_{i,j,k}\mathbf{M}_{i,j}\otimes\mathbf{N}_{j,k}}$. It follows that $\mathbf{M}\otimes_B\mathbf{N} \cong {}_A\mathbf{M}_B\mathbf{N}_C$.

□

Corollary 3.2.11. *Horizontal composition of modules is defined in any symmetric monoidal dagger category with biproducts.*

Proof. This is true because in lemma 3.2.10, we gave a construction of the horizontal composition of any two modules, in a symmetric monoidal dagger category $\mathbf{C}$ with biproducts. □

Lemma 3.2.12. *Let* $\mathbf{C}$ *be a symmetric monoidal dagger category with dagger biproducts,* $2(\mathbf{C})(A,B) \cong \mathbf{C}^{nk}$*, where* $A = \oplus_{i=1}^n I_i$*,* $B = \oplus_{j=1}^k I_j$*.*

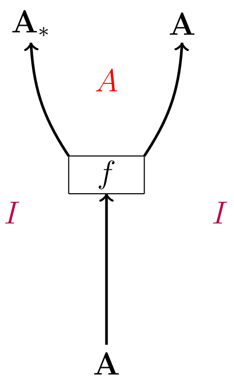
Proof. $2(\mathbf{C})(A,B) \cong \oplus_{i,j} 2(\mathbf{C})(I_i, I_j)$, as the hom-set of a biproduct is isomorphic to the biproduct of the hom-sets. This is isomorphic to $\oplus_{i,j}\mathbf{C}$ by lemma 3.2.9. □

3.3 Interpretation of $2(\mathbf{C})$

So far we have discussed the motivation for $2(\mathbf{C})$ and its mathematical details, but not yet how we can use this category as a model for quantum computing. Now we will discuss how classical and quantum information and their interaction can be modelled in $2(\mathbf{C})$.

Quantum systems are modules of the unit object, and classical systems are modules of the corresponding classical structure on itself. We will see in section 3.6, that this corresponds to the situation in $FHilb$.

A measurement on a module $\mathbf{M}$ is a 2-cell from a quantum module to a classical module.



(3.25)

The set of possible measurement outcomes is given by the classical structure represented by the area marked with A. The set of possible resulting states is the module $\mathbf{A}$. Every state a of the measurement outcome A determines the corresponding resulting state, $\mathbf{A}_a \otimes {}_a\mathbf{A}$. In $2(FHilb)$, every possible measurement outcome a corresponds to a projector which determines the component of $\mathbf{A}$, which is the resulting state. Later we will prove that the outcome of every measurement corresponds to a special dagger Frobenius algebra.

The difference between this model and the models we described earlier is that measurements are not morphisms that destroy the state, but morphisms that explicitly describe the correspondence between the resulting state and the measurement outcome. As a result, the model describes all possible measurement outcomes with all corresponding resulting states at once. This is all in correspondence with the many worlds interpretation. We will prove in section 3.5, that measurements in the hom-category $2(\mathbf{C})(I, I)$ are unitary. This seems problematic because one of the characteristics of quantum mechanics is that measurements are not reversible. However, we should keep in mind that this model regards physical systems from an outside perspective and that it contains all information about all possible measurement outcomes. When we do an actual experiment, our perspective is from the inside, as we are only able to access the information corresponding to our measurement outcomes.

For quantum computation, this is particularly useful because it allows us to talk about quantum protocols containing measurements, despite the uncertainty of the measurement outcomes. The corresponding diagrammatic language exhibits the topological properties of quantum protocols, which are independent of implementations of the protocol. This gives us a high-level understanding of the structure of these protocols. We will not discuss these topological properties in this dissertation. Further information and discussion can be found in [20].

3.4 Properties of 2(-)

We will demonstrate that for any monoidal dagger category $\mathbf{C}$, $2(\mathbf{C})$ is a dagger 2-category. Moreover, if $\mathbf{C}$ has biproducts, all hom-categories of $2(\mathbf{C})$ have biproducts. Finally we will show that the construction $2(-)$ preserves compactness. We suspect that the 2- category $2(\mathbf{C})$ is a symmetric monoidal 2-category, like $FHilb$. This is a 2-category with an additional tensor product on 0-, 1- and 2-cells, and with symmetry, unit, and associator natural isomorphisms, as well as the necessary coherence conditions. The definition of a symmetric monoidal 2-category is given in [15]. It will take us too far to explore this in this dissertation.

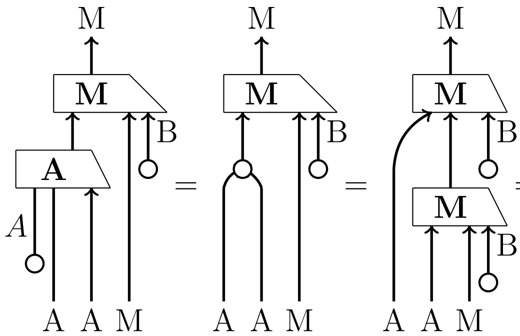
3.4.1 2-category

We will prove that for any symmetric monoidal dagger category $\mathbf{C}$, where all coequalisers exist, and where all coequalisers are dagger coequalisers, $2(\mathbf{C})$ is a 2-category. First, we will prove some lemmas concerning composition in $2(\mathbf{C})$.

Lemma 3.4.1. *Let $\mathbf{C}$ be a symmetric monoidal dagger category, where all coequalisers exist, and where all coequalisers are dagger coequalisers. Every 0-cell $(A, \triangledown_A)$ of $2(\mathbf{C})$ has an identity 1-cell for horizontal composition. This is the 1-cell $A \otimes A \otimes A \xrightarrow{\mathbf{A}} A$, of which he left and right module action of $\mathbf{A}$ are given by $\triangledown_A$. This 1-cell satisfies the equations $\mathbf{A} \otimes_A \mathbf{M} \cong \mathbf{A}$ and $\mathbf{N} \otimes_A \mathbf{A} \cong \mathbf{A}$, for any two 1-cells $\mathbf{M} \in 2(\mathbf{C})(A, B)$ and $\mathbf{N} \in 2(\mathbf{C})(C, A)$, and for any two 0-cells B and C.*

Proof. We will prove that $\mathbf{A} \otimes_{\mathbf{A}} \mathbf{M} \cong \mathbf{M}$, by showing that $A \otimes M \xrightarrow{{}_A\mathbf{M}} M$ is the coequaliser of $\mathbf{A}_A \otimes \mathrm{id}_M$ and $\mathrm{id}_A \otimes {}_A\mathbf{M}$, and that this defines a module homomorphism ${}_A\mathbf{A} \otimes \mathbf{M}_A \xrightarrow{{}_A\mathbf{M}} \mathbf{M}$. Then we will prove that the isomorphism exists, using the uniqueness property of coequalisers. The other isomorphism is proven in a similar way, by the coequaliser $\mathbf{N}_A$. This proof is very similar to that of lemma 3.2.7.

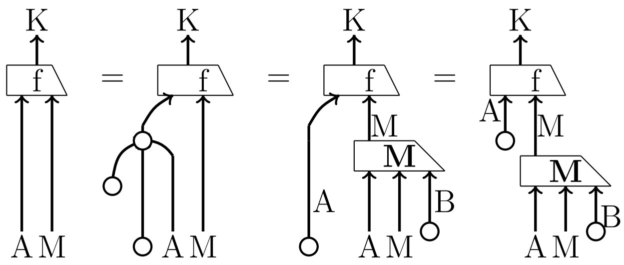
First, we need to show that ${}_A\mathbf{M} \circ (\mathbf{A}_A \otimes \mathrm{id}_M) = {}_A\mathbf{M} \circ (\mathrm{id}_A \otimes {}_A\mathbf{M})$. This is shown in the picture below.

${}_A\mathbf{M} \circ (\ \mathbf{A}_A \otimes id_M) =$  $= {}_A\mathbf{M} \circ (id_A \otimes {}_A\mathbf{M})$ (3.26)

The second equality holds because the multiplication of $\mathbf{A}$ is defined to be $\bigtriangledown_A$. The third equality holds by equation 3.4, and by the Frobenius laws.

It is left to show, that whenever we have a morphism $f : A \otimes M \to K$, for some object $K \in \mathbf{C}$, it factors through ${}_A\mathbf{M}$.

It follows from the picture, that any such f equals $f \circ (u_A \otimes \mathrm{id}_M) \circ {}_A\mathbf{M}$:

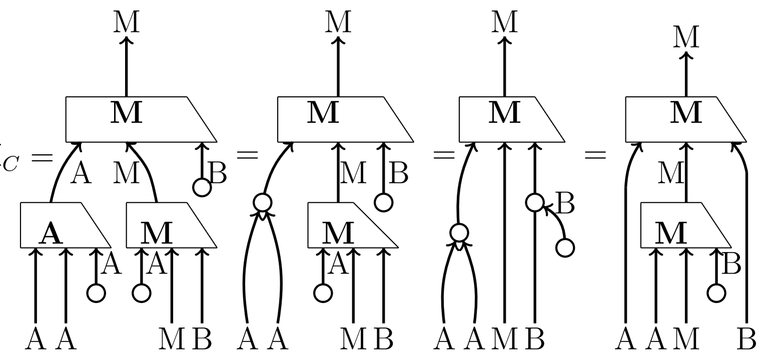


(3.27)

The first equality is by equation the Frobenius laws, the second because we have assumed that $f \circ (\mathbf{A}_A \otimes \mathrm{id}_M) = f \circ (\mathrm{id}_A \otimes {}_A\mathbf{M})$.

So we have proven that $A \otimes M \xrightarrow{{}_A\mathbf{M}} M$ is the required coequaliser.

We need to verify that ${}_A\mathbf{M}$ is a module homomorphism from ${}_A\mathbf{A} \otimes \mathbf{M}_B$ to $\mathbf{M}$. In that case, $\mathbf{M}$ satisfies equation 3.10. This is shown in the diagram below.

${}_A\mathbf{M} \circ {}_A\mathbf{A} \otimes \mathbf{M}_C =$  $= \mathbf{M} \circ {}_A\mathbf{M}$ (3.28)

The second equality holds because $\mathbf{A}$ equals $\bigtriangledown_A$, the third and the fourth hold by equation 3.4 and the Frobenius laws.

So we have the commuting triangles

$$
\begin{array}{ccc}
\mathbf{A}\otimes\mathbf{M} & \xrightarrow{\pi} & \mathbf{A}\otimes_A\mathbf{M} \\
& {}_A\mathbf{M}\searrow & \downarrow l_{\mathbf{M}} \\
& & \mathbf{M}
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{N}\otimes\mathbf{A} & \xrightarrow{\pi} & \mathbf{N}\otimes_A\mathbf{A} \\
& \mathbf{N}_A\searrow & \downarrow r_{\mathbf{N}} \\
& & \mathbf{N}
\end{array}
\quad , \tag{3.29}
$$

where $l_{\mathbf{M}}$ and $r_{\mathbf{N}}$ are the canonical morphisms, by uniqueness of coequalisers. These are $2-cells$ because π, $\mathbf{M}_A$, and ${}_A\mathbf{N}$ are epic module homomorphisms. Furthermore, $l_{\mathbf{M}}$ and $r_{\mathbf{N}}$ are unitary because coequalisers are epic. □

In the diagrammatic language, we often do not draw the identity module $\mathbf{A}$ on A because this is implicit in the area that represents the 0-cell A. Instead we sometimes draw a dashed line or nothing.

Lemma 3.4.2. *Let* $\mathbf{C}$ *be a symmetric monoidal dagger category, where all coequalisers exist, and where all coequalisers are dagger coequalisers. Horizontal composition of 1-cells in* $2(\mathbf{C})$ *is associative.*

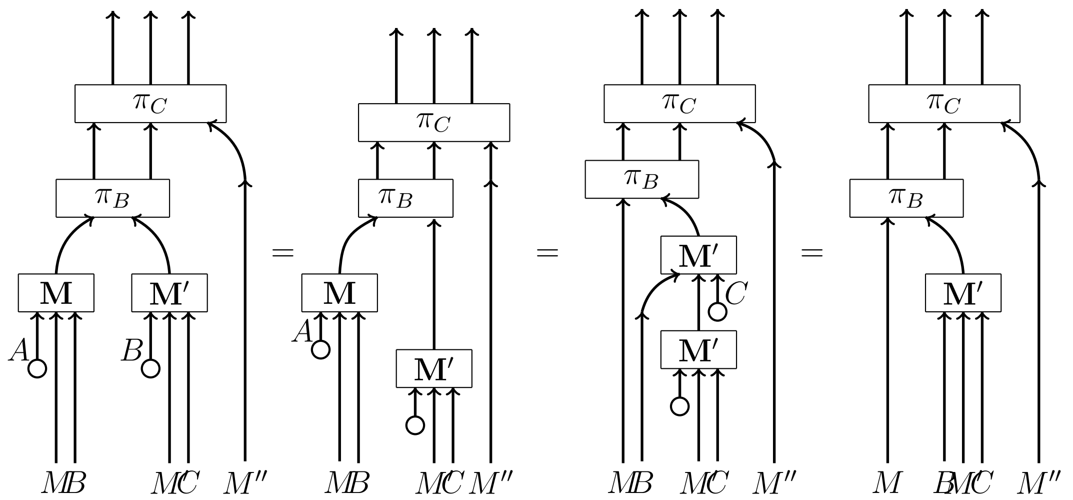
Proof. There is a canonical isomorphism $(\mathbf{M}\otimes_B\mathbf{M}')\otimes_C\mathbf{M}'' \xrightarrow{\bar{\alpha}} \mathbf{M}\otimes_B(\mathbf{M}'\otimes_C\mathbf{M}'')$ because both expressions are the image of coequalisers of the morphisms $M\otimes B\otimes M'\otimes C\otimes M'' \xrightarrow{f_i} M\otimes M'\otimes M''$ in $\mathbf{C}$, for $i = 1,..,4$. These morphisms are defined as

- $f_1 = \mathbf{M}_B\otimes\mathbf{M}'_C\otimes\mathrm{id}_{M''}$
- $f_2 = \mathbf{M}_B\otimes\mathrm{id}_{M'}\otimes\, {}_C\mathbf{M}''$
- $f_3 = \mathrm{id}_M\otimes\, {}_B\mathbf{M}'\otimes\, {}_C\mathbf{M}''$
- $f_4 = \mathrm{id}_M\otimes\, {}_B\mathbf{M}'_C\otimes\mathrm{id}_{M''}$

We will prove this for $(\mathbf{M}\otimes_B\mathbf{M}')\otimes_C\mathbf{M}''$. The proof for $\mathbf{M}\otimes_B(\mathbf{M}'\otimes_C\mathbf{M}'')$ is very similar.

The coequaliser is given by the morphism $M\otimes M'\otimes M'' \xrightarrow{\pi_C\circ(\pi_B\otimes\mathrm{id}_{M''})} (M\otimes_B M')\otimes_C M''$ in $\mathbf{C}$, where π_B is the coequaliser of $\mathbf{M}_B\otimes\mathrm{id}_{M'}$ and $\mathrm{id}_M\otimes\, {}_B\mathbf{M}'$, and π_C is the coequaliser of $(\mathbf{M}\otimes_{\mathbf{B}}\mathbf{M}')_C\otimes\mathrm{id}_{M''}$ and $\mathrm{id}_{M\otimes_B M'}\otimes\, {}_C\mathbf{M}''$.

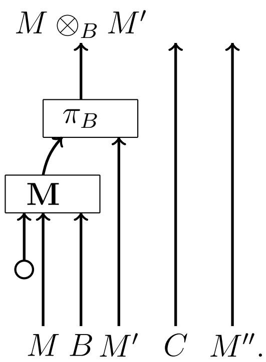
Because π_B is a coequaliser, $\pi_C\circ(\pi_B\otimes\mathrm{id}_{M''})\circ f_2 = \pi_C\circ(\pi_B\otimes\mathrm{id}_{M''})\circ f_3$. Also, $\pi_C\circ(\pi_B\otimes\mathrm{id}_{M''})\circ f_1 = \pi_C\circ(\pi_B\otimes\mathrm{id}_{M''})\circ f_4$. The first statement is straightforward, The last statement is illustrated by the picture below.



(3.30)

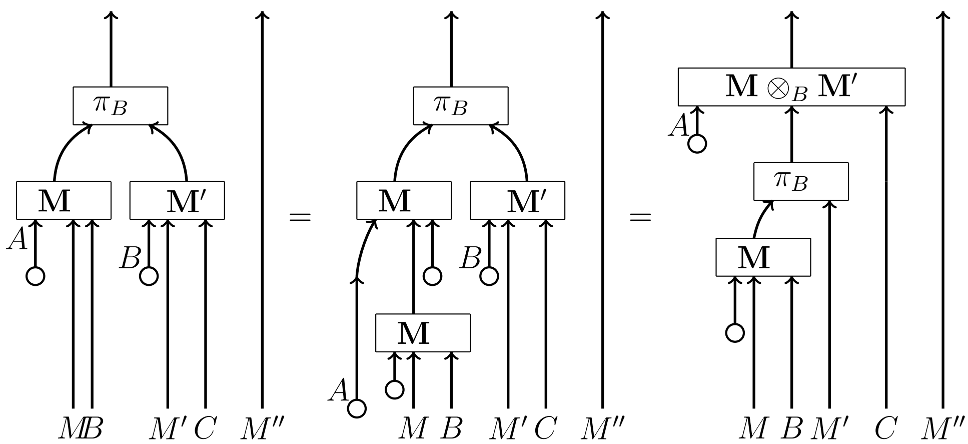
$\pi_C \circ (\pi_B \otimes \mathrm{id}_{M''}) \circ f_1$ equals the expression on the left and $\pi_C \circ (\pi_B \otimes \mathrm{id}_{M''}) \circ f_4$ equals the expression on the right. The second equality is because π_B is a coequaliser, the third equality is by 3.4.

Furthermore, if we postcompose f_1 and f_2 with $\pi_b \otimes \mathrm{id}_{M''}$, we get $((\mathbf{M} \otimes_B \mathbf{M}')_C \otimes \mathrm{id}_{M''}) \circ h$ and $(\mathrm{id}_{M\otimes_B M'} \otimes\ {}_C\mathbf{M}'') \circ h$ respectively, for the composite function

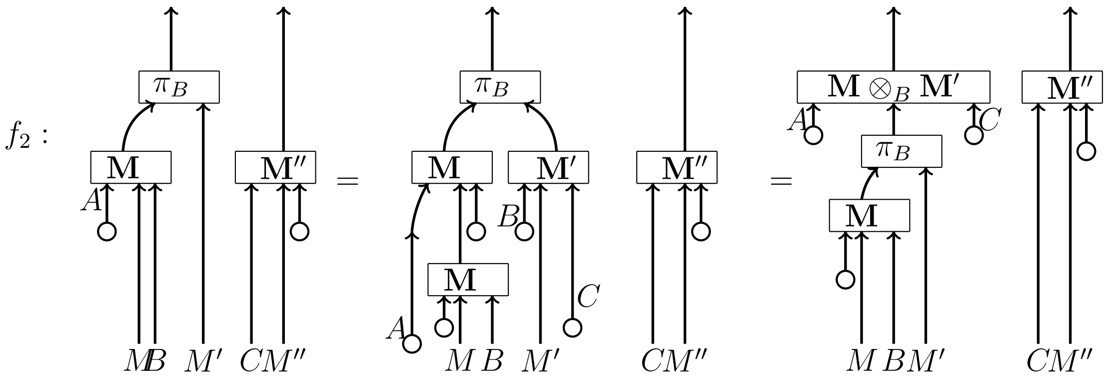
$h :=$ 

(3.31)

This is shown in the diagrams below:

$f_1 :$ 

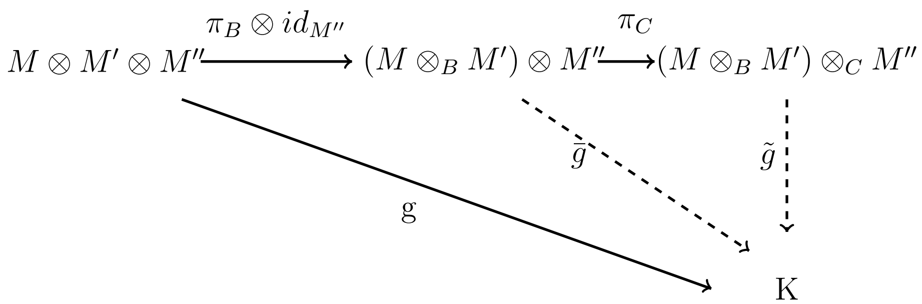
(3.32)



(3.33)

The last equality of both equations holds because π_B is a module homomorphism. It follows that $\pi_C \circ (\pi_B \otimes \mathrm{id}_{M''}) \circ f_1 = \pi_C \circ (\pi_B \otimes \mathrm{id}_{M''}) \circ f_2$. This is sufficient for the proof that $\pi_C \circ (\pi_B \otimes \mathrm{id}_{M''}) \circ f_i = \pi_C \circ (\pi_B \otimes \mathrm{id}_{M''}) \circ f_j$, for all $i, j \in \{1, ..., 4\}$

If there is another map $M \otimes M' \otimes M'' \xrightarrow{g} K$ in $\mathbf{C}$, such that $g \circ f_1 = g \circ f_2 = g \circ f_3 = g \circ f_4$, it factors through $(M \otimes_B M') \otimes_C M''$, by the following diagram:

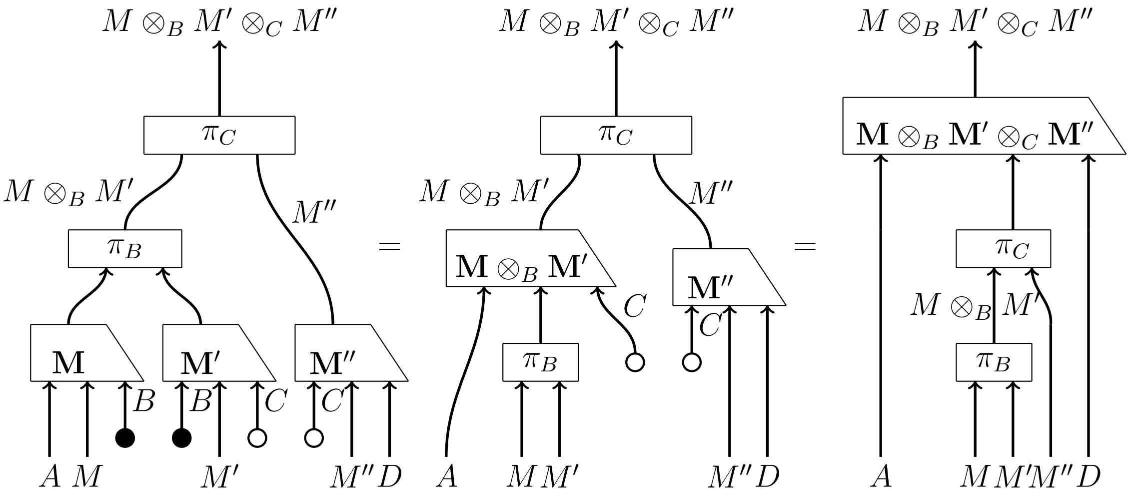


(3.34)

The first diagram commutes as $\pi_B \otimes \mathrm{id}_{M''}$ is the coequaliser of f_2 and f_3 and the second diagram commutes as π_C is the coequaliser of $\pi_B \otimes \mathrm{id}_{M''} \circ f_i$ for $\{i = 1, 2\}$.

It is left to show that ${}_A\mathbf{M} \otimes \mathrm{id}_{M'} \otimes \mathbf{M}''_C \xrightarrow{\pi_C \circ (\pi_B \otimes \mathrm{id}_{M''})} {}_A(\mathbf{M} \otimes_B \mathbf{M}' \otimes_C \mathbf{M}'')_C$ is a module, and therefore equation 3.10 holds.

This follows from the following picture, which is just unfolding the definitions of $\mathbf{M} \otimes_B \mathbf{M}'$ and $(\mathbf{M} \otimes_B \mathbf{M}') \otimes_C \mathbf{M}''$.



□

Lemma 3.4.3. *Module homomorphisms in a category* **C** *are closed under composition*

Proof. Let $\mathbf{M}, \mathbf{N}, \mathbf{K}$ be module homomorphisms of A on M, N, K respectively, and let $\mathbf{M} \xrightarrow{f} \mathbf{N}$ and $\mathbf{N} \xrightarrow{g} \mathbf{K}$ be module homomorphisms. By definition, the equalities $f \circ \mathbf{M} = \mathbf{N} \circ (\mathrm{id}_A \otimes f)$ and $g \circ \mathbf{N} = \mathbf{K} \circ (\mathrm{id}_A \otimes g)$ hold. It follows that $(g \circ f) \circ \mathbf{M} = g \circ \mathbf{N} \circ (\mathrm{id}_A \otimes f) = \mathbf{K} \circ (\mathrm{id}_A \otimes g) \circ (\mathrm{id}_A \otimes f) = \mathbf{K} \circ (\mathrm{id}_A \otimes (g \circ f))$, hence $(g \circ f)$ is a module homomorphism. We used associativity of morphisms in **C**. □

Lemma 3.4.4. *Let* $\mathbf{M}, \mathbf{M}'$ *and* $\mathbf{M}''$ *be right modules of* B *and let* $\mathbf{N}, \mathbf{N}'$ *and* $\mathbf{N}''$ *be left modules of* B*. Let* $M \xrightarrow{f} M' \xrightarrow{f'} M''$ *and* $N \xrightarrow{g} N' \xrightarrow{g'} N''$ *be module homomorphisms. the following equality holds:*

$$(f' \circ f) \otimes_B (g' \circ g) = (f' \otimes_B g') \circ (f \otimes_B g) \tag{3.35}$$

This is called the interchange law.

Proof. Analogous to our argument for definition 3.2.5, all squares of the two diagrams below commute.

$$\begin{array}{ccccc}
M \otimes B \otimes N & \underset{\mathrm{id}_M \otimes {}_B\mathbf{N}}{\overset{\mathbf{M}_B \otimes \mathrm{id}_N}{\rightrightarrows}} & M \otimes N & \xrightarrow{\pi} & M \otimes_B N \\
\downarrow{\scriptstyle f \otimes \mathrm{id}_B \otimes g} & & \downarrow{\scriptstyle f \otimes g} & & \downarrow{\scriptstyle f \otimes_B g} \\
M' \otimes B \otimes N' & \underset{\mathrm{id}'_N \otimes {}_B\mathbf{N}'}{\overset{\mathbf{M}'_B \otimes \mathrm{id}_{N'}}{\rightrightarrows}} & M' \otimes N' & \xrightarrow{\pi'} & M' \otimes_B N' \\
\downarrow{\scriptstyle f' \otimes \mathrm{id}_B \otimes g'} & & \downarrow{\scriptstyle f' \otimes g'} & & \downarrow{\scriptstyle f' \otimes_B g'} \\
M'' \otimes B \otimes N'' & \underset{\mathrm{id}''_M \otimes {}_B\mathbf{N}''}{\overset{\mathbf{M}''_B \otimes \mathrm{id}_{N''}}{\rightrightarrows}} & M'' \otimes N'' & \xrightarrow{\pi''} & M'' \otimes_B N''
\end{array} \tag{3.36}$$

$$
\begin{array}{ccccc}
M \otimes B \otimes N & \overset{\mathbf{M}_B \otimes \mathrm{id}_N}{\underset{\mathrm{id}_N \otimes\ _B\mathbf{N}}{\rightrightarrows}} & M \otimes N & \xrightarrow{\pi} & M \otimes_B N \\
\Big\downarrow {\scriptstyle (f' \circ f) \otimes \mathrm{id}_B \otimes (g' \circ g)} & & \Big\downarrow {\scriptstyle (f' \circ f) \otimes (g' \circ g)} & & \vdots\, {\scriptstyle (f' \circ f) \otimes_B (g' \circ g)} \\
M'' \otimes B \otimes N'' & \overset{\mathbf{M}''_B \otimes \mathrm{id}_{N''}}{\underset{\mathrm{id}''_M \otimes\ _B\mathbf{N}''}{\rightrightarrows}} & M'' \otimes N'' & \xrightarrow{\pi''} & M'' \otimes_B N''
\end{array}
\tag{3.37}
$$

Because the interchange law holds for the tensor product in $\mathbf{C}$, the two outer squares and the arrow in the middle coincide. It follows that $(f' \circ f) \otimes_B (g' \circ g)$ and $(f' \otimes_B g') \circ (f \otimes_B g)$ must be equal by uniqueness of coequalisers. □

Theorem 3.4.5. *Let* $\mathbf{C}$ *be a monoidal dagger category where all coequalisers exist, and where all coequalisers are dagger coequalisers.* $2(\mathbf{C})$ *is a weak 2-category.*

Proof. We will prove that $2(\mathbf{C})$ satisfies each of the conditions of a weak 2-category, as stated in definition 3.1.1:

1 Horizontal composition is as we described it before in 3.2.4 in terms of coequalisers. We assume that all coequalisers in $\mathbf{C}$ exist and that these are dagger modules, so modules and module homomorphisms are closed under horizontal composition. Module homomorphisms are closed under vertical composition by 3.4.3.

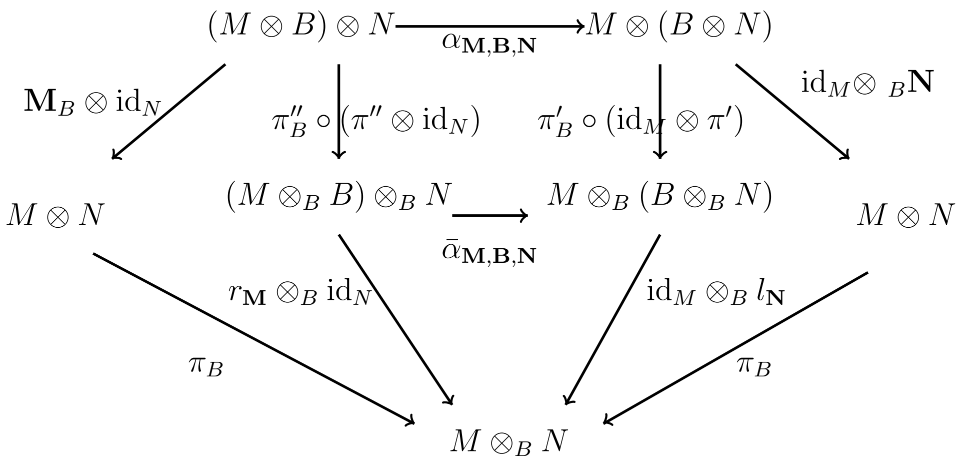
2 By lemma 3.4.2 there exists a unitary 2-cell $\alpha_{\mathbf{M},\mathbf{M}',\mathbf{M}''}$:$(\mathbf{M} \otimes_B \mathbf{M}') \otimes_C \mathbf{M}'' \Rightarrow \mathbf{M} \otimes_B (\mathbf{M}' \otimes_C \mathbf{M}'')$ for every triple of 1-cells $A \xrightarrow{\mathbf{M}} B \xrightarrow{\mathbf{M}'} C \xrightarrow{\mathbf{M}''} D$.

3 By lemma 3.4.1, for each 0-cell A there exists an identity module $\mathbf{A}$, such that for every 1-cell $A \xrightarrow{\mathbf{M}} B$ there are two invertible 2-cells $l_{\mathbf{M}} : \mathbf{A} \otimes_A \mathbf{M} \Rightarrow \mathbf{M}$ and $r_{\mathbf{M}} : \mathbf{M} \otimes_B \mathbf{B} \Rightarrow \mathbf{M}$.

4 The 2-cells are module homomorphisms in $\mathbf{C}$, and their vertical composition is given by composition in $\mathbf{C}$, which is associative. By lemma 3.4.3, 2-cells are closed under vertical composition. The identity morphisms on 1-cells in $2(\mathbf{C})$ are the identity morphisms on the underlying objects in $\mathbf{C}$.

5 The interchange law follows from lemma 3.4.4.

Naturality of l, r and $\bar{\alpha}$ follows from carefully checking all commutative diagrams that are needed to define l, r, and $\bar{\alpha}$. The triangle and pentagon equality follow from the triangle and pentagon equality for the monoidal category $\mathbf{C}$ and the definition of horizontal composition by coequalisers. We will omit the proof of the pentagon equality, as it is just a matter of checking definitions carefully. The proof of the triangle equality is given below:



(3.38)

In this diagram, $\pi'', \pi', \pi''_B, \pi'_B$, and π_B are the coequalisers that are used to define the horizontal composition of the appropriate modules.

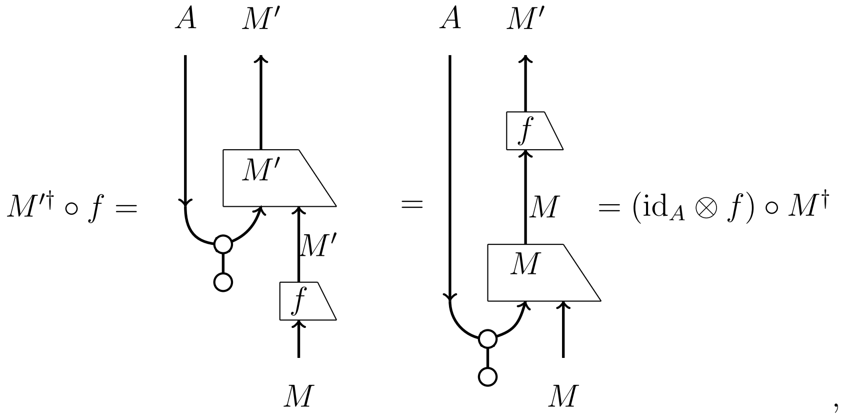
The outer square commutes as π_B is the coequaliser of $\mathbf{M}_B \otimes \mathrm{id}_N$ and $\mathrm{id}_M \otimes\, _B\mathbf{N}$. The upper square commutes by definition of α in 3.4.2. The left and right rectangles commute by definition of l and r 3.29 and horizontal composition 3.2.5. It follows that the lower triangle in the middle, which is the triangle equality, commutes as well. □

3.4.2 Dagger

Lemma 3.4.6. *Let* $\mathbf{C}$ *be a symmetric monoidal dagger category, where all coequalisers exist, and where all coequalisers are dagger coequalisers. The category* $2(\mathbf{C})$ *is a dagger 2-category.*

Proof. Given a dagger functor $\dagger : \mathbf{C} \to \mathbf{C}$, we can define the involutive endofunctor $\ddagger : 2(\mathbf{C}) \to 2(\mathbf{C})$, which is the identity on 0-cells and 1-cells, and on 2-cells it is defined as $\ddagger(f) := \dagger f$. We will prove that this is well-defined.

If $f : \mathbf{M} \to \mathbf{M'}$ is a module homomorphism between the modules $A \otimes M \xrightarrow{\mathbf{M}} M$ and $A \otimes M' \xrightarrow{\mathbf{M'}} M'$ in $\mathbf{C}$, then f corresponds to a morphism $f : M \to M'$ in $\mathbf{C}$ that satisfies the equation 3.7. It follows that



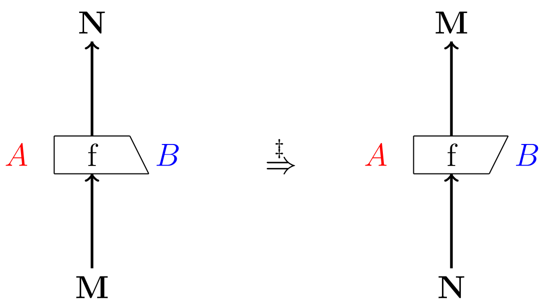
(3.39)

by the definition of dagger modules, 3.6.

If we apply $\dagger$ to both sides of the equation, we get $(M'^{\dagger} \circ f)^{\dagger} = ((\mathrm{id}_A \otimes f) \circ M^{\dagger})^{\dagger}$, which equals $f^{\dagger} \circ M' = M \circ (\mathrm{id}_A \otimes f^{\dagger})$. It follows that $\ddagger f = \dagger f : M' \to M$ satisfies equation 3.7, so it is a module homomorphism from $\mathbf{M'}$ to $\mathbf{M}$.

This proves that 2-cells are closed under $\ddagger$. Furthermore, $\ddagger$ is the identity on 0- and 1-cells, so it defines a dagger functor on $2(\mathbf{C})$. $\square$

As in monoidal categories, the dagger is represented in the diagrammatic language as

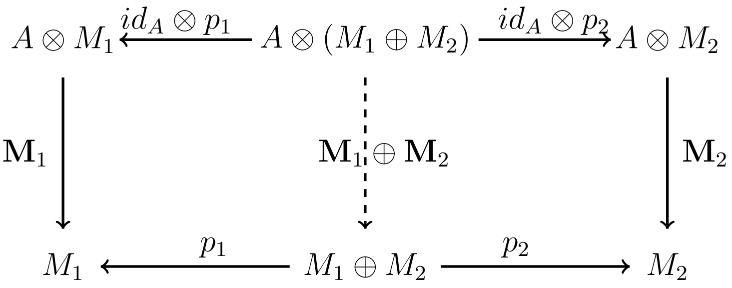


(3.40)

3.4.3 Biproducts

Lemma 3.4.7. *Let* $\mathbf{C}$ *be a monoidal dagger category with biproducts where all coequalisers exist, where all coequalisers are dagger coequalisers, and where the tensor distributes over the biproducts. The hom-categories* $2(\mathbf{C})(A, B)$ *have dagger biproducts for all 0-cells* $A, B \in 2(\mathbf{C})$.

Proof. Any biproduct $(M_1 \oplus M_2, p_1, p_2, i_1, i_2)$ defines a product by exercise 2.5.2 of [9]. As a consequence, for every two modules $\mathbf{M}_i : A \otimes M_i \to M_i$, $i = 1, 2$, regarded as morphisms in **C**, there exists a unique morphism $\mathbf{M}_1 \oplus \mathbf{M}_2 : A \otimes (M_1 \oplus M_2) \to (M_1 \oplus M_2)$, which makes the diagram below commute.

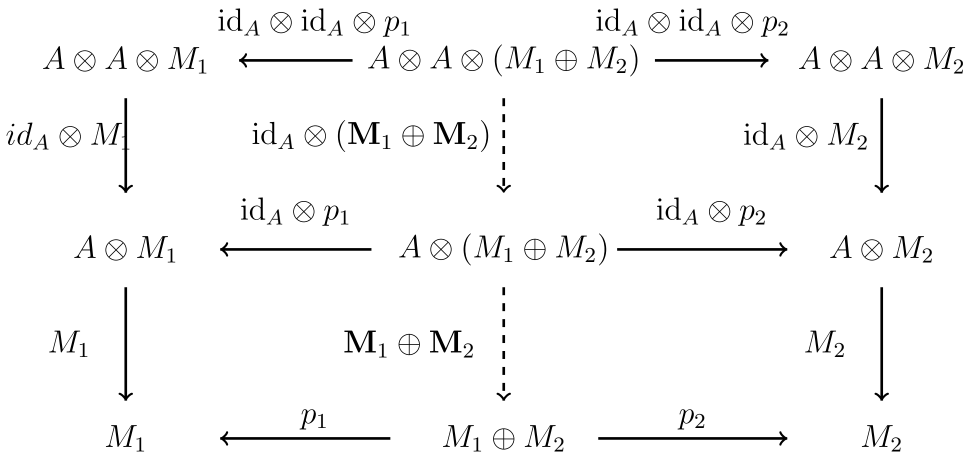


(3.41)

This unique morphism is given by the product $\mathbf{M}_1 \times \mathbf{M}_2$ in **C**. By commutativity of the diagram, we have $p_i \circ \mathbf{M}_1 \oplus \mathbf{M}_2 = \mathbf{M}_i \circ (\mathrm{id}_A \otimes p_i)$, so p_1 and p_2 are module homomorphisms. It follows that this is also true for i_1 and i_2, since $\mathbf{M}_1 \oplus \mathbf{M}_2 \circ (\mathrm{id}_A \otimes i_i) = i_i \circ p_i \circ (\mathbf{M}_1 \oplus \mathbf{M}_2) \circ (\mathrm{id}_A \otimes i_i) = i_i \circ \mathbf{M}_i \circ (\mathrm{id}_A \otimes p_i) \circ (\mathrm{id}_A \otimes i_i) = i_i \circ \mathbf{M}_i$.

We need to show that $\mathbf{M}_1 \oplus \mathbf{M}_2$ is a module.

We will show by the commuting diagrams below that $\mathbf{M}_1 \oplus \mathbf{M}_2$ satisfies equation 3.4. The right hand side of this equation is the unique arrow that makes the first diagram below commute, the left hand side is the unique arrow that makes the second diagram below commute, and the outside squares of the two diagrams are equal by 3.4. It follows from the uniqueness property of products that the two arrows must be equal.



(3.42)

$$
\begin{array}{ccccc}
A \otimes A \otimes M_1 & \xleftarrow{\mathrm{id}_A \otimes \mathrm{id}_A \otimes p_1} & A \otimes A \otimes (M_1 \oplus M_2) & \xrightarrow{\mathrm{id}_A \otimes \mathrm{id}_A \otimes p_2} & A \otimes A \otimes M_2 \\
\downarrow {\scriptstyle \bigtriangledown_A \otimes \mathrm{id}_{M_1}} & & \vdots {\scriptstyle \bigtriangledown_A \otimes (id_{M_1} \oplus id_{M_2})} & & \downarrow {\scriptstyle \bigtriangledown_A \otimes \mathrm{id}_{M_1}} \\
A \otimes M_1 & \xleftarrow{\mathrm{id}_A \otimes p_1} & A \otimes (M_1 \oplus M_2) & \xrightarrow{\mathrm{id}_A \otimes p_2} & A \otimes M_2 \\
\downarrow {\scriptstyle \mathbf{M}_1} & & \vdots {\scriptstyle \mathbf{M}_1 \oplus \mathbf{M}_2} & & \downarrow {\scriptstyle \mathbf{M}_2} \\
M_1 & \xleftarrow{p_1} & M_1 \oplus M_2 & \xrightarrow{p_2} & M_2
\end{array}
\tag{3.43}
$$

The equations 3.5 and 3.6 are proven in the same way, with the appropriate diagrams.

It is straightforward that $\mathbf{M}_1 \oplus \mathbf{M}_2$ is in fact a biproduct. The following equations are satisfied because $(M_1 \oplus M_2, p_1, p_2, i_1, i_2)$ is a biproduct in $\mathbf{C}$:

$$\mathrm{id}_{M_1} = p_1 \circ i_1 \tag{3.44}$$
$$\mathrm{id}_{M_2} = p_2 \circ i_2 \tag{3.45}$$
$$0_{M_1,M_2} = p_{M_2} \circ p_{M_1} \tag{3.46}$$
$$0_{M_2,M_1} = p_{M_1} \circ p_{M_2} \tag{3.47}$$
$$\mathrm{id}_{M_1 \oplus M_2} = (i_{M_1} \circ p_{M_1} + i_{M_2} \circ p_{M_2}) \tag{3.48}$$

If $(M_1 \oplus M_2, p_1, p_2, i_1, i_2)$ is a dagger biproduct, the equation below makes $\mathbf{M}_1 \oplus \mathbf{M}_2$ into a dagger biproduct because it is a dagger biproduct in $\mathbf{C}$.

$$i_i^\dagger = p_i \tag{3.49}$$

The proof of the lemma follows from the fact that every bimodule $\mathbf{M} : A \otimes M \otimes B \to M$ corresponds to a module $\mathbf{M'} : (A \otimes B) \otimes M \to M$. □

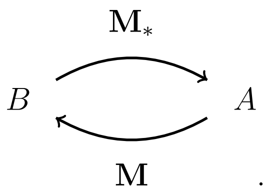
A sufficient condition for the existence of biproducts in $2(\mathbf{C})$ is that $\mathbf{C}$ is a symmetric monoidal dagger compact category with biproducts, where all coequalisers exist, and where all coequalisers are dagger coequalisers. In chapter 2 we have seen that in this case the tensor distributes over the biproduct.

The proofs we have seen so far in this section only concern 1-cells and 2-cells. These proofs still hold when we restrict the 0-cells to special dagger Frobenius algebras. This means that $2(\mathbf{C})_s$ is a dagger 2-category, and if $\mathbf{C}$ has biproducts, all hom-sets of $2(\mathbf{C})_s$ have biproducts.

3.4.4 Compactness

By the axioms for 2-categories, if we regard the 0-cells as categories, then the 1-cells define functors between 0-cells and the 2-cells define natural transformations. We use this to define compactness in 2-categories.

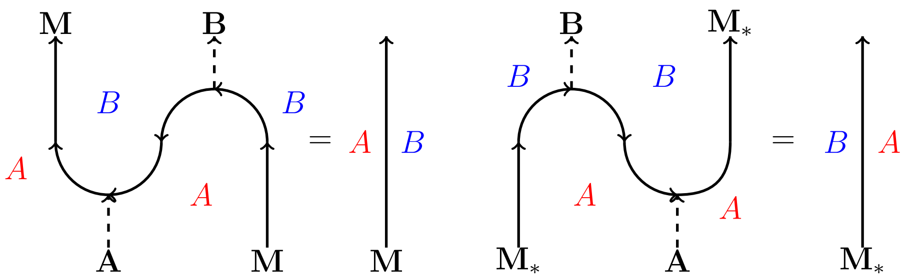
A 1-cell $A \xrightarrow{\mathbf{M}} B$ in a 2-category has a dual $B \xrightarrow{\mathbf{M}_*} A$, if there exists and adjunction



(3.50)

.

This is when there are natural isomorphisms between the hom-sets $A(\mathbf{M}_*(b), a)$ and $B(b, \mathbf{M}(a))$ for all objects a of A and b of B. In that case, there exist natural transformations $\bar{\eta} : \mathbf{B} \Rightarrow \mathbf{M}_* \otimes_A \mathbf{M}$ and $\bar{\epsilon} : \mathbf{M} \otimes_B \mathbf{M}_* \Rightarrow \mathbf{A}$, such that
$\mathbf{M} \otimes_B \mathbf{B} \xRightarrow{\mathrm{id}_{\mathbf{M}} \otimes_B \bar{\eta}} \mathbf{M} \otimes_B \mathbf{M}_* \otimes_A \mathbf{M} \xRightarrow{\bar{\epsilon} \otimes_A \mathrm{id}_{\mathbf{M}}} \mathbf{A} \otimes_A \mathbf{M} \cong \mathrm{id}_{\mathbf{M}}$, and
$\mathbf{B} \otimes_B \mathbf{M}_* \xRightarrow{\bar{\eta} \otimes_B \mathrm{id}_{\mathbf{M}_*}} \mathbf{M}_* \otimes_A \mathbf{M} \otimes_B \mathbf{M}_* \xRightarrow{\mathrm{id}_{\mathbf{M}_*} \otimes_A \bar{\epsilon}} \mathbf{M}_* \otimes_A \mathbf{A} \cong \mathrm{id}_{\mathbf{M}_*}$ Graphically, these equations look like

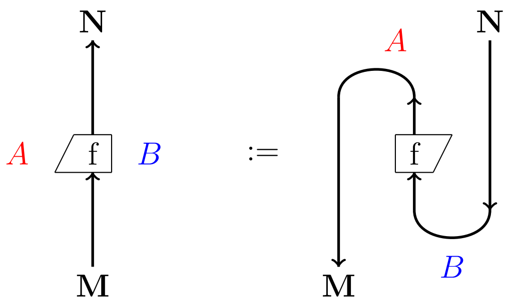


(3.51)

Definition 3.4.8. A 2-category is compact when every 1-cell $\mathbf{M}$ has a dual 1-cell $\mathbf{M}_*$

In the diagrammatic language of a compact 2-category, we can bend all wires without changing the meaning of the diagram as long as the start and end point stays fixed.

In a dagger compact 2-category, the dual $\mathbf{M}_* \overset{f_*}{\Rightarrow} \mathbf{N}_*$ of the 2-cell $\mathbf{M} \overset{f}{\Rightarrow} \mathbf{N}$ is given by the 2-cell



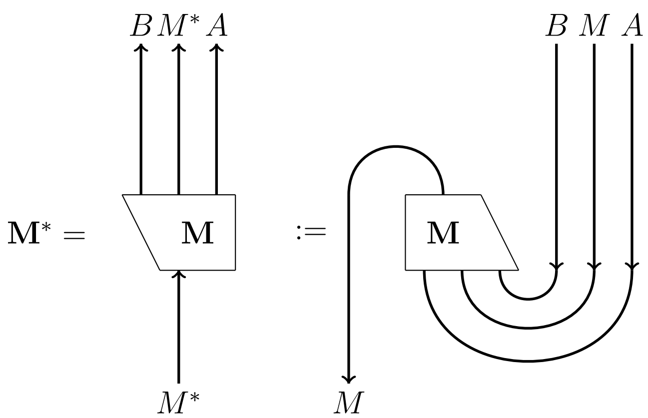
(3.52)

If a 2-category has only one 0-cell, this definition of compactness is exactly the definition of compactness of monoidal categories.

The fact that monoidal categories are a special case of 2-categories, gives us a good reason to believe that the diagrammatic language of 2-categories that we have discussed so far is complete and sound. This is just an extension of the diagrammatic language of monoidal categories and the proof of coherence of this diagrammatic language could most likely be extended to a proof of coherence for the 2-categorical diagrammatic language. However, this would involve a lot of work, and there are many variations of 2-categories. For this reason, Selinger omits such proofs in [16].

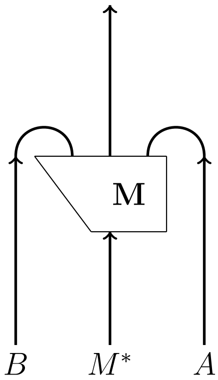
Lemma 3.4.9. *Let* $\mathbf{C}$ *be a symmetric monoidal dagger compact category, where all coequalisers exist, and where all coequalisers are dagger coequalisers, then the category* $2(\mathbf{C})_s$ *is compact.*

Proof. If $\mathbf{C}$ is compact, every module $\mathbf{M} : A \otimes M \otimes B \to M$ has a dual morphism $\mathbf{M}^*$. In diagrammatic language this is defined as



(3.53)

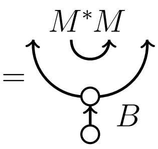
Definition 3.4.10. In the category $2(\mathbf{C})$, the dual of a 1-cell $\mathbf{M} \in 2(\mathbf{C})(A, B)$ is given by the 1-cell $\mathbf{M}_* \in 2(\mathbf{C})(B, A)$ defined as

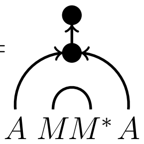
$\mathbf{M}_* :=$ 

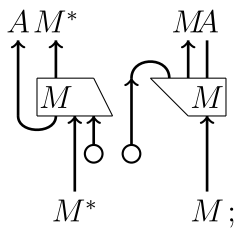
(3.54)

Observe that $\mathbf{M}_* = \mathbf{M}^{\dagger *}$, by 3.6, so this is the conjugation of $\mathbf{M}$.

The 2-cells $\bar{\eta}_M$ and $\bar{\epsilon}_M$ are defined as $\bar{\eta}_M := {}_B\mathbf{M}_* \otimes_A \mathbf{M}_B \circ \pi \circ f$ and $\bar{\epsilon}_M := \tilde{g} \circ {}^A\mathbf{M} \otimes_B \mathbf{M}_*^A$, where

$f :=$ 

$g :=$ 

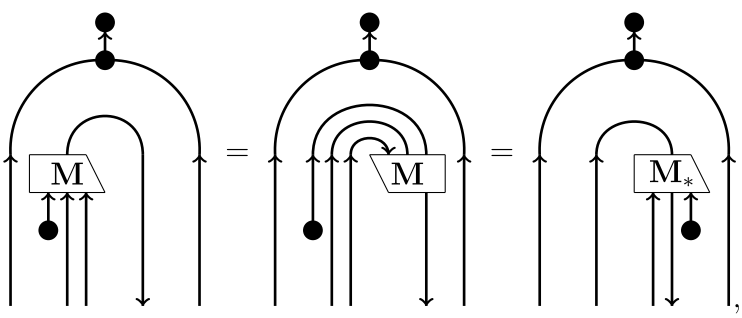
${}^A\mathbf{M} \otimes_B \mathbf{M}_*^A :=$ 

(3.55)

π is the coequaliser of $g \circ (\mathrm{id}_A \otimes \mathbf{M}_B \otimes \mathrm{id}_{M^* \otimes A})$ and $g \circ (\mathrm{id}_{A \otimes M} \otimes {}_B\mathbf{M}_* \otimes \mathrm{id}_A)$; $\tilde{g}$ is the unique morphism that makes the diagram below commute

$$\begin{array}{ccc} A \otimes M \otimes M^* \otimes A & \xrightarrow{\pi} & A \otimes M \otimes_B M^* \otimes A \\ & \searrow^{g} & \downarrow \tilde{g} \\ & & \mathrm{A} \end{array} \quad . \tag{3.56}$$

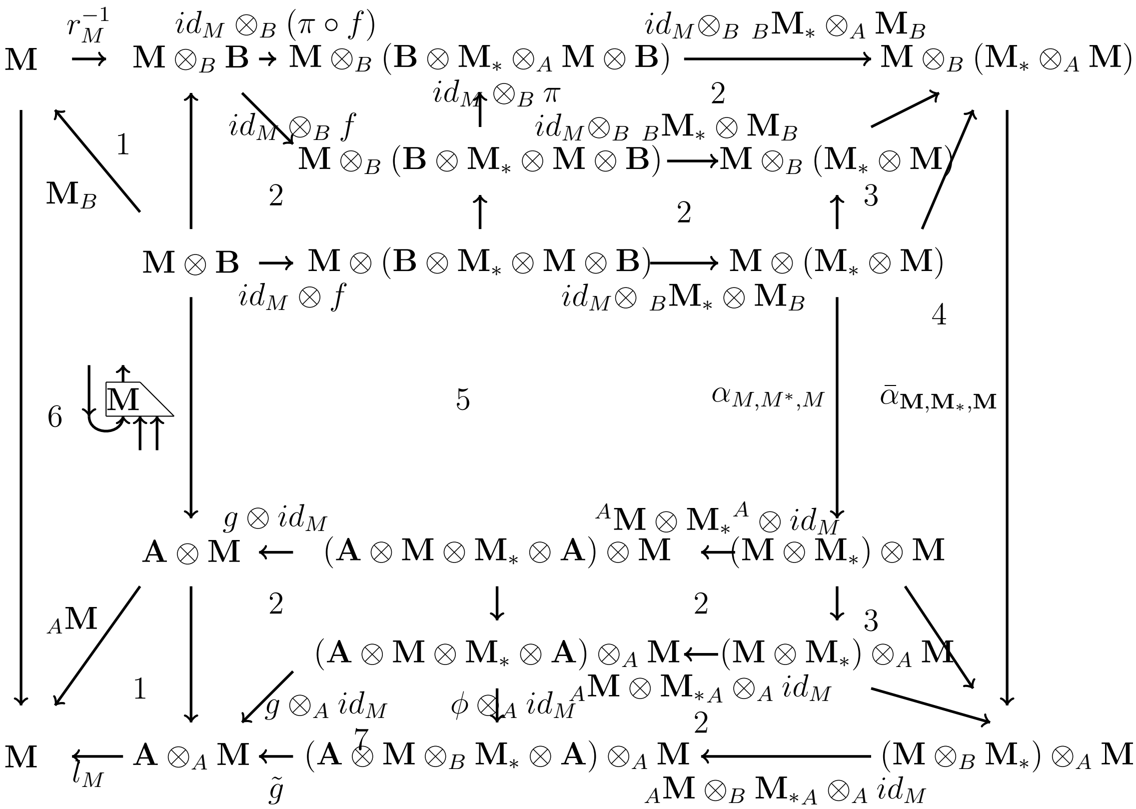
The morphism $\tilde{g}$ exists by definition of the coequaliser because $g \circ (\mathrm{id}_A \otimes \mathbf{M}_B \otimes \mathrm{id}_{M^* \otimes A})$ equals



(3.57)

which equals $g \circ (\mathrm{id}_{A\otimes M} \otimes\ {}_B\mathbf{M}_* \otimes \mathrm{id}_A)$.

We will prove that the diagram below commutes, and therefore, that the equality $l_{\mathbf{M}} \circ (\bar{\epsilon}_{\mathbf{M}} \otimes_A \mathrm{id}_{\mathbf{M}}) \circ \bar{\alpha} \circ (\mathrm{id}_{\mathbf{M}} \otimes_B \bar{\eta}_{\mathbf{M}}) \circ r_{\mathbf{M}}^{-1} = \mathrm{id}_{\mathbf{M}}$ holds. This equality corresponds to the outside of the diagram. The other equality is proven in a similar way.



(3.58)

To keep the diagram as clear as possible, we omitted the labels of all coequalisers, unless we needed them for an equation; so the empty arrows correspond to the appropriate coequaliser.

We will prove commutativity of all numbered diagrams:

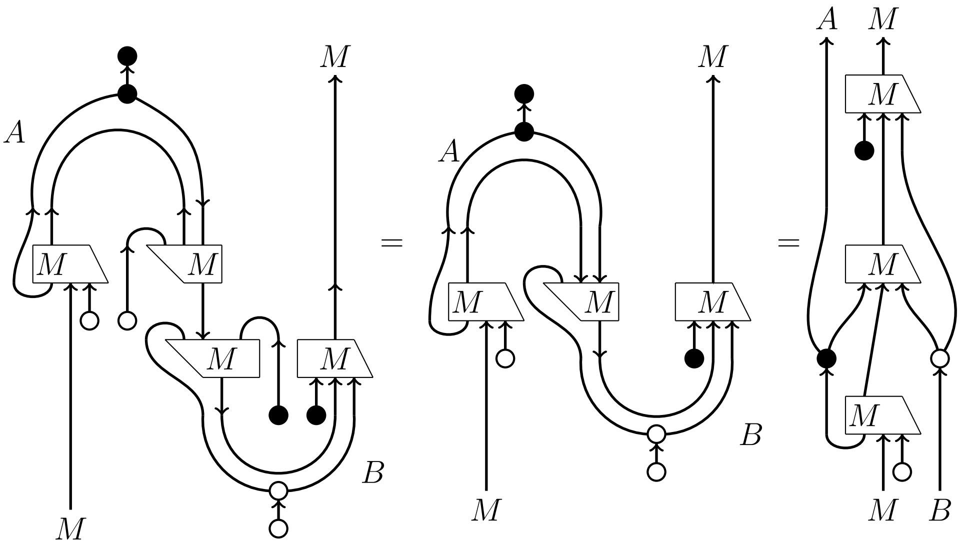
1 By the identity condition, lemma 3.4.1

2 By definition of horizontal composition of morphisms, 3.2.5

3 This composition of coequalisers is analogous to what we did in lemma 3.4.2

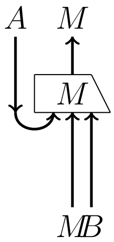
4 By the associativity of horizontal composition, lemma 3.4.2

5 From the upper left corner to the lower left corner in clockwise direction, this is



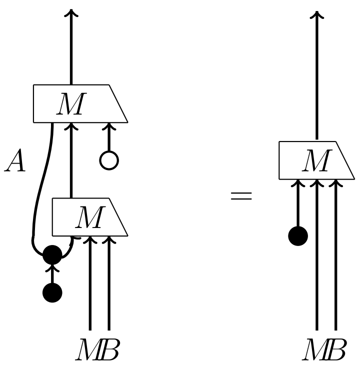
(3.59)

The equalities are given by the definition of modules, the Frobenius laws and specialness of A and B. This equals the counter clockwise direction:



(3.60)

6 The composition of morphisms from the upper right corner to the lower left corner, in clockwise direction, is given by the left hand side of the equation below; and the composition of morphisms in the counter clockwise direction is given by the right hand side. The equality follows from the axioms of modules, the Frobenius laws and specialness of A.



(3.61)

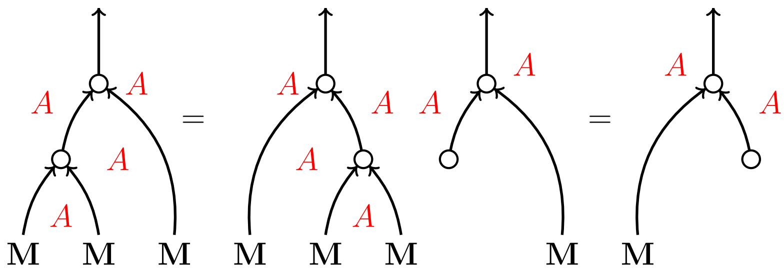
7 By definition of $\tilde{g}$

□

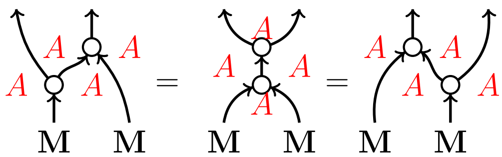
3.4.5 Frobenius algebras in 2(C)

We will define the notion of dagger Frobenius algebras in 2-categories. The main result of this section is the existence of canonical dagger Frobenius algebras of certain 1-cells.

Definition 3.4.11. Let $2\mathbf{C}$ be a dagger weak 2-category. A Frobenius algebra in $2\mathbf{C}$ is a 1-cell $\mathbf{M} \in 2\mathbf{C}(A, A)$, for some 0-cell A, together with two module homomorphisms $u_M : \mathbf{A} \to \mathbf{M}$ and $\bigtriangledown_M : \mathbf{M} \to \mathbf{M} \otimes_A \mathbf{M}$, that satisfy the equations below.

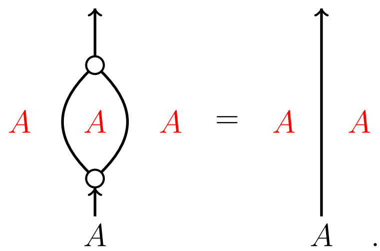


(3.62)



(3.63)

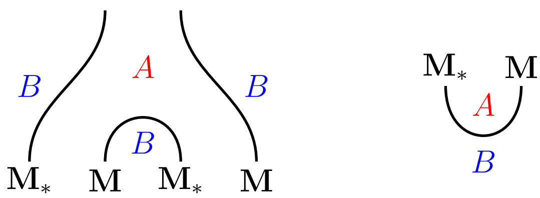
A Frobenius algebra in $2\mathbf{C}$ is special if it satisfies



(3.64)

We will prove in the next lemma that the outcome of any measurement has a canonical dagger Frobenius algebra.

Lemma 3.4.12. *Let $2\mathbf{C}$ be a dagger compact weak 2-category. Every 1-cell of the form $\mathbf{M}_* \otimes_A \mathbf{M}$ has a canonical dagger Frobenius algebra, given by $u_{\mathbf{M}_* \otimes_A \mathbf{M}} := \bar{\eta}_{\mathbf{M}}$, $\bigtriangledown_{\mathbf{M}_* \otimes_A \mathbf{M}} := \mathrm{id}_{\mathbf{M}_*} \otimes_A \bar{\eta}^{\dagger}_{\mathbf{M}_*} \otimes_A \mathrm{id}_{\mathbf{M}}$, and their daggers.*



(3.65)

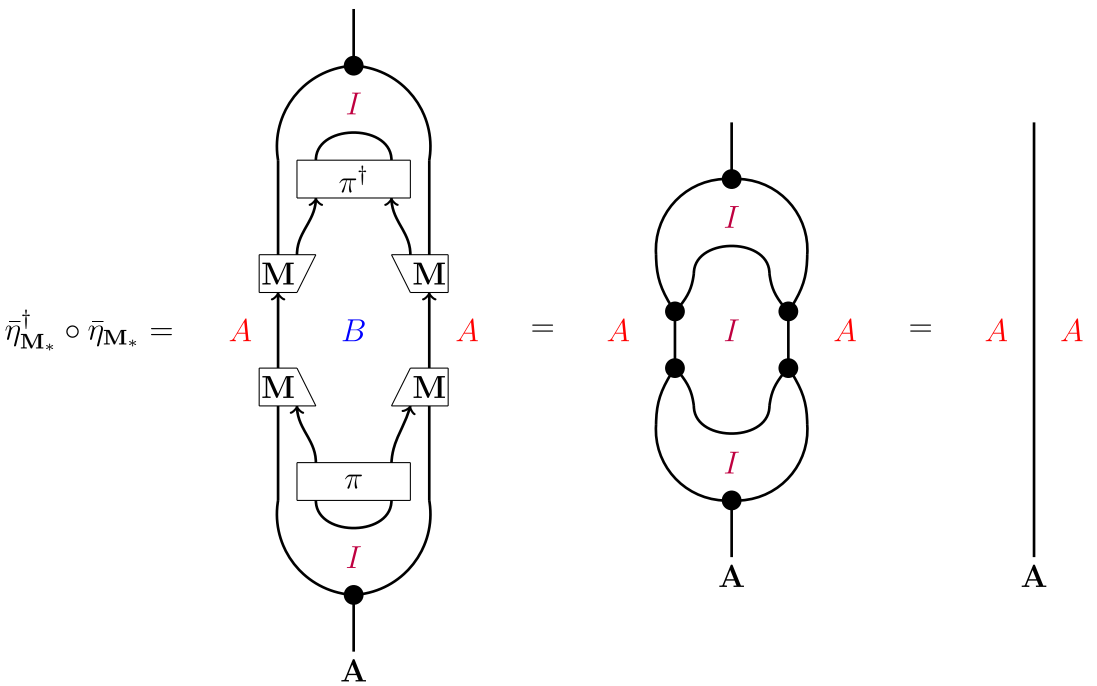
Proof. The Frobenius laws hold because we can bend any wire in the way we want without changing the meaning of the diagram, due to compactness of $2\mathbf{C}$. □

This dagger Frobenius algebra is special, when $\bigtriangledown_{\mathbf{M}_*\otimes_A\mathbf{M}} \circ \Delta_{\mathbf{M}_*\otimes_A\mathbf{M}} = \mathrm{id}_{\mathbf{M}_*\otimes_\mathbf{A}\mathbf{M}}$. This is the case when $\bar{\eta}^\dagger_{\mathbf{M}_*} \circ \bar{\eta}_{\mathbf{M}_*} = \mathrm{id}_{\mathbf{M}_*\otimes_A\mathbf{M}}$, so when $\bar{\eta}_{\mathbf{M}_*}$ is an isometry.

We will give a sufficient condition, under which a canonical dagger Frobenius algebra is special.

Lemma 3.4.13. *Let $2\mathbf{C}$ be a dagger compact weak 2-category. The canonical dagger Frobenius algebra of a 1-cell $\mathbf{M}_* \otimes_A \mathbf{M} \in 2\mathbf{C}(B, B)$ is special, if $B = I$, $\mathbf{M}_A = \mathbf{A}_A$, and A is special.*

Proof. By the diagram below, $\bar{\eta}_{\mathbf{M}_*}$ is unitary, which implies that the canonical Frobenius algebra on $\mathbf{M}_* \otimes_A \mathbf{M}$ is special.



(3.66)

The first equality is a matter of unfolding the definition of $\bar{\eta}$. The second equality holds because in this case, the coequaliser π is the identity on $\mathbf{A} \otimes \mathbf{A}$, and ${}_A\mathbf{M}_* \otimes \mathbf{M}_A = \bigtriangledown_A \otimes \bigtriangledown_A$. The third equality follows from the Frobenius laws and specialness of A.

□

3.5 2-categorical models with canonical dagger Frobenius algebras

The quantum protocols described in [20], i.e. quantum teleportation, require the existence of a deletion map on physical systems. Therefore, it is useful to have a category that has a deletion map for all 1-cells. In this section, we will define a category in which all 1-cells have a canonical dagger Frobenius algebra, and consequently, a deletion map.

Definition 3.5.1. Let $2\mathbf{C}$ be a dagger compact weak 2-category. A symmetric 1-cell of $2\mathbf{C}$ is a 1-cell of the form $\mathbf{M}_* \otimes_A \mathbf{M} \in 2\mathbf{C}(B, B)$ for some 1-cell $\mathbf{M} \in 2\mathbf{C}(A, B)$.

Definition 3.5.2. Let $2\mathbf{C}$ be a dagger compact weak 2-category. The category $Sym(2\mathbf{C})$ is the subcategory of $2\mathbf{C}$, which consists of the 0-cells of $2\mathbf{C}$, all horizontal compositions of symmetric 1-cells of $2\mathbf{C}$, and all 2-cells of $2\mathbf{C}$, between 1-cells of $Sym(2\mathbf{C})$.

Lemma 3.5.3. *Let* $\mathbf{C}$ *be a symmetric monoidal dagger compact category, where all coequalisers exist. The category* $Sym(2(\mathbf{C})_s)$ *is a full subcategory of* $2(\mathbf{C})_s$*; moreover, it is a dagger compact weak 2-category.*

Proof. First of all, $Sym(2(\mathbf{C}))$ has the same 0-cells as $2(\mathbf{C})$. All 1-cells of $Sym(2(\mathbf{C}))$ are 1-cells of $2(\mathbf{C})$. By definition, $Sym(2(\mathbf{C}))$ is closed under horizontal composition. Every identity 1-cell $\mathbf{A}$ is by definition isomorphic to $\mathbf{A} \otimes_A \mathbf{A}$, which is symmetric. This means that $Sym(2(\mathbf{C}))$ contains an identity 1-cell for each 0-cell. By definition, we have the following equality for all symmetric 1-cells $\mathbf{M}, \mathbf{N} \in 2(\mathbf{C})(A, A)$ for some A: $Sym(2(\mathbf{C}))(\mathbf{M}, \mathbf{N}) = 2(\mathbf{C})(\mathbf{M}, \mathbf{N})$. It follows that $Sym(2(\mathbf{C}))$ contains an identity 2-cell for every 1-cell, and the 2-cells of $Sym(2(\mathbf{C}))$ are closed under vertical and horizontal composition. In addition, $Sym(2(\mathbf{C}))$ inherits $\bar{\alpha}$, l, r, the interchange law, and the triangle and pentagon equalities. Moreover, $Sym(2(\mathbf{C}))$ is a weak 2-category.

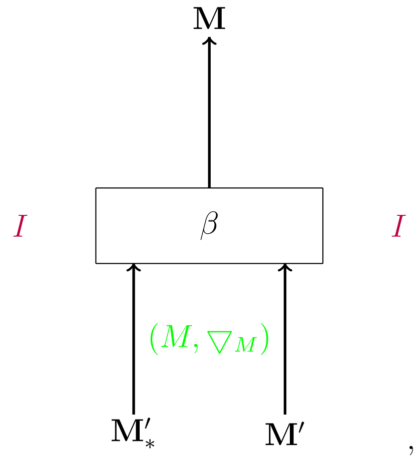
The dagger functor is the identity on 0-cells and 1-cells. The dagger of a 2-cell between a horizontal composition of symmetric 1-cells is again a 2-cell between a horizontal composition of symmetric 1-cells. As a result, $Sym(2(\mathbf{C}))$ is a dagger 2-category.

All symmetric 1-cells have canonical dagger Frobenius algebras by lemma 3.4.12; therefore, they are self-dual. Every 1-cell of $2(\mathbf{C})$, which is a horizontal composition of symmetric 1-cells of $2(\mathbf{C})$, is dual to the horizontal composition of the same 1-cells, but in reversed order. As a consequence, $Sym(2(\mathbf{C}))$ is compact. □

The next lemma proves that all non-demolition measurements in $2\mathbf{C}(I, I)$ are unitary.

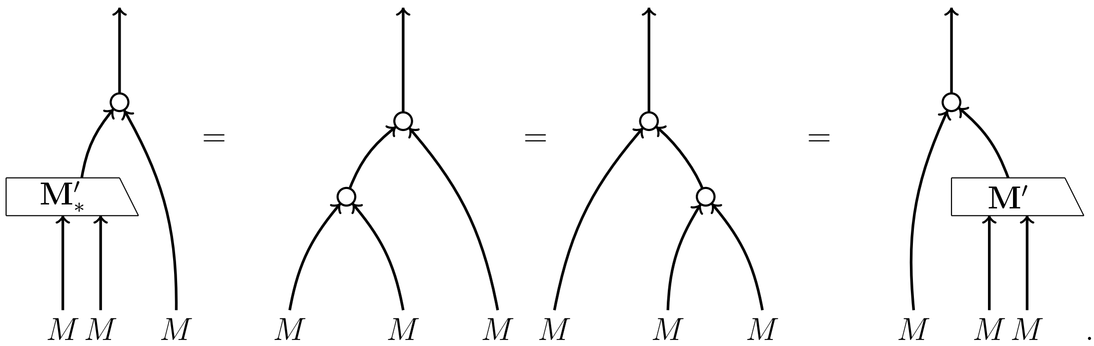
Lemma 3.5.4. *Let* $2\mathbf{C}$ *be dagger weak 2-category. Every 1-cell* $\mathbf{M} \in 2\mathbf{C}(I, I)$ *is isomorphic to the 1-cell* $\mathbf{M}'_* \otimes_{(M,\bigtriangledown_M)} \mathbf{M}'$*, where* $(M, \bigtriangledown_M)$ *is a commutative dagger Frobenius algebra on* M *in* $\mathbf{C}$*, and* $\mathbf{M}' = \bigtriangledown_M$*.*

Proof. We will prove that there exists a unitary 2-cell



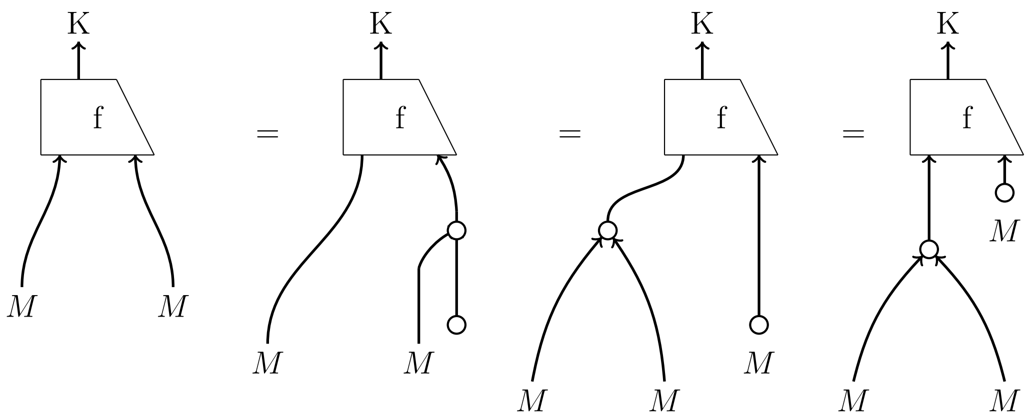
(3.67)

Observe that $\triangledown_M$ defines a coequaliser of $\mathbf{M}'_{*M} \otimes \mathrm{id}_M$ and $\mathrm{id}_M \otimes\ {}_M\mathbf{M}'$. This is because



(3.68)

Furthermore, any morphism $M \otimes M \xrightarrow{f} K$, such that $f \circ (\mathbf{M}'_{*M} \otimes \mathrm{id}_M) = f \circ (\mathrm{id}_M \otimes\ {}_M\mathbf{M}')$, factors through $\triangledown_M$. This is shown in the following diagram, where we write $\triangledown_M$ for both $\mathbf{M}'_{*M}$ and ${}_M\mathbf{M}_*$.



(3.69)

The first equality is by the Frobenius laws, the second by our assumption that $f \circ (\mathbf{M}'_{*M} \otimes \mathrm{id}_M) = f \circ (\mathrm{id}_M \otimes\ {}_M\mathbf{M}')$.

The coequaliser $\bigtriangledown_M$ is a module homomorphism, ${}_I\mathbf{M}'_* \otimes \mathbf{M}'_I \xrightarrow{\bigtriangledown_M} \mathbf{M}$ because $\mathbf{M}$ is the unique module, such that equation 3.10 is satisfied. This is simply because $\mathbf{M}$ is a module of the identity 0-cel, and these module are multiplications by a unitary scalar.

Let π be the coequaliser that defines the module $\mathbf{M}'_* \otimes_{(M,\bigtriangledown_M)} \mathbf{M}'$. It follows from the definition of a coequaliser, that there is a canonical morphism $(M \otimes_{(M,\bigtriangledown_M)} M) \xrightarrow{\beta} M$. By uniqueness of coequalisers, we have the following equality: $\beta^\dagger \circ \beta \circ \pi = \mathrm{id}_{M\otimes_{(M,\bigtriangledown_M)}M} \circ \pi$. All coequalisers are epic, so it follows that $\beta^\dagger \circ \beta = \mathrm{id}_{M\otimes_{(M,\bigtriangledown_M)}M}$. Similarly, $\beta \circ \beta^\dagger = \mathrm{id}_{\mathbf{M}}$, so β is unitary. By lemma 3.2.9, β corresponds to a unitary 1-cell $(\mathbf{M}'_* \otimes_{(M,\bigtriangledown_M)} \mathbf{M}') \xrightarrow{\beta} \mathbf{M}$

□

A consequence of this lemma, is that all scalars of $2\mathbf{C}$ are isomorphic to symmetric 1-cells.

Corollary 3.5.5. *Let $2\mathbf{C}$ be dagger weak 2-category. The hom-categories of scalars $Sym(2(\mathbf{C}))(I,I)$ and $2(\mathbf{C})(I,I)$ are isomorphic.*

Proof. By lemma 3.5.4, every 1-cell of $2(\mathbf{C})$ is isomorphic to a symmetric 1-cell. It follows that $Sym(2(\mathbf{C}))$ is isomorphic to $2(\mathbf{C})$. □

Corollary 3.5.6. *Let $2\mathbf{C}$ be a dagger compact weak 2-category. Every 1-cell in $2\mathbf{C}(I,I)$, as defined in 3.4.12, is isomorphic to a special dagger Frobenius algebra.*

Proof. By lemma 3.5.4, every symmetric 1-cell $\mathbf{M}$ is isomorphic to $\mathbf{M}'_* \otimes_{(M,\bigtriangledown_M)} \mathbf{M}'$, where $\mathbf{M}' = \bigtriangledown_M$. This module is special by lemma 3.4.13. □

3.6 2(FHilb)

In this section, we will explore the structure of the category $2(FHilb)$ and we will compare it to $2Hilb$, the standard 2-categorical model for quantum theory.

3.6.1 Modules on dagger Frobenius algebras in $FHilb$

In chapter 2, we discussed dagger Frobenius algebras in $FHilb$. First, we will recall the important results. Next, we will explore the structure of modules on dagger Frobenius algebras in $FHilb$. Finally, we will look at the 2-category $2(FHilb)$.

By [19], all dagger Frobenius algebras in $FHilb$ correspond to finite dimensional C^* algebras. Normal and special dagger Frobenius algebras correspond to normal C^*-algebras. These C^*-algebras correspond to biproducts of matrix algebras. The unit of an n-dimensional matrix algebra A is the state $I \to A$ that maps 1 to the n-dimensional unit matrix. For 1-dimensional matrix algebras, this unit is the only copyable state. All commutative matrix algebras in $FHilb$ are 1-dimensional matrices. Consequently, commutative C^* algebras are biproducts of 1-dimensional matrix algebras. The copyable states of a commutative C^* algebra A are sums of the copyable states of its summands. It is shown in [9] that these copyable states form an orthonormal basis of A. If this C^*-algebra is normal, this is an orthonormal basis.

A module $\mathbf{M}$ on a special dagger Frobenius algebra A in $FHilb$ corresponds to a decomposition of the underlying Hilbert space M, into the orthogonal subspaces $\mathbf{M}_{u_i}(M)$. The states u_i are the units of the matrix algebras. This is true by lemma 3.2.10 because $FHilb$ is a category with biproducts, and the units of the biproducts are copyable states. By this lemma, the automorphisms $\mathbf{M}_{u_i}$ are projections.

As we described in section 3.3, a quantum state is a module of the unit object. In $FHilb$, this corresponds to a system that is not decomposed into orthogonal subspaces. A classical state is a module of a classical structure on itself, where the module action equals the multiplication map of the classical structure. In $FHilb$, this corresponds to a decomposition of a Hilbert space in 1-dimensional subspaces. This is exactly what we expect.

3.6.2 Horizontal composition in $Fhilb$

We will show that modules on dagger Frobenius algebras in $FHilb$ are R-modules. C^*-algebras are rings, and Hilbert spaces are Abelian groups with respect to $+$. The R-module action rx of an element r of the C^* algebra on an element x of the Hilbert space, is given by the module $\mathbf{M}(r,x)$. The equalities $r(x+y) = rx + ry$ and $(r+s)x = rx + rs$ hold, as $\mathbf{M}$ is a morphism in $FHilb$, so it is a linear map. The equality$(rs)x = r(sx)$ follows from axiom 3.4. The equality $1x = x$ follows from 3.5.

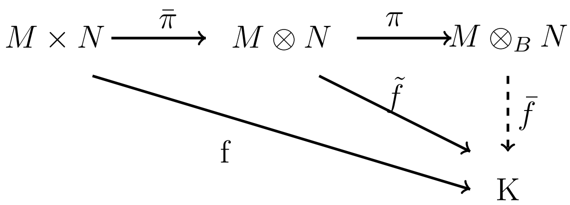
Normally, the tensor product $M \otimes_B M'$ of a right R-module M, and a left R-module M', over a ring B, is the defined as an Abelian group with a universal bilinear map $\pi_B : M \times N \to M \otimes_B N$, with respect to the ring B. This means that for every bilinear map $M \times M' \xrightarrow{f} N$ with respect to B, there is a unique map $\bar{f} : M \otimes_B M' \to N$, such that $\bar{f} \circ \pi_B = f$.

We will prove that the tensor product $\mathbf{M} \otimes_B \mathbf{M'}$ of modules in $FHilb$, corresponds to the normal definition of a tensor product of R-modules. For any map $M \otimes M' \xrightarrow{g} N$, $g \circ (\mathbf{M}_B \otimes \mathrm{id}_{M'}) = g \circ (\mathrm{id}_M \otimes\ _B\mathbf{M'})$, if and only if $g(x \cdot b, x') = g(x, b \cdot x')$, for all elements $x \in M$, $x' \in M'$, and $b \in B$. Furthermore, all morphisms in $FHilb$ are linear maps. It follows that the collection of bilinear maps $g : M \otimes M' \to N$ with respect to B corresponds exactly to the collection of maps g, such that $g \circ (\mathbf{M}_b \otimes \mathrm{id}_{M'}) = g \circ (\mathrm{id}_M \otimes \mathbf{M}_b)$ for all $b \in B$.

In other words, the coequaliser $M \otimes_B M'$ defines a universal arrow, in the sense that for every bilinear map $\tilde{f} : M \otimes M' \to N$ with respect to the module action of B, there is a unique map $\bar{f} : M \otimes_B M' \to N$ such that $\bar{f} \circ \pi = \tilde{f}$.

The tensor product $M \otimes N$, corresponds to the tensor product of Hilbert spaces over the unit object, the ring $\mathbb{C}$. This is defined by a universal bilinear map $M \times N \xrightarrow{\bar{\pi}} M \otimes N$, with respect to $\mathbb{C}$. We will prove that the composition of the two universal arrows, $M \times N \xrightarrow{\pi \circ \bar{\pi}} M \otimes_B N$, is a universal arrow for bilinear maps with respect to B. If π_B and $\pi \circ \bar{\pi}$ are both universal, they must be isomorphic.

Universality of $\pi \circ \bar{\pi}$ follows from the diagram below.



(3.70)

Any bilinear map $f : M \times N \to K$ with respect to B is bilinear with respect to $\mathbb{C}$. By universality of $\bar{\pi}$, there is a unique map $\tilde{f} : M \otimes N \to K$, such that $\tilde{f} \circ \bar{\pi} = f$. The map $\bar{\pi}$ is not bilinear with respect to B, unless $B = \mathbb{C}$, so $\tilde{f}$ must be bilinear with respect to B. By universality of π, there is a unique map $\bar{f} : M \otimes_B N \to K$, such that $\bar{f} \circ \pi = \tilde{f}$. It follows that $\bar{f}$ is the unique map, such that $\bar{f} \circ \pi \circ \bar{\pi} = f$. In the case that $B = \mathbb{C}$, there is nothing to prove.

3.6.3 $2FHilb$

We will prove that the category $2(FHilb_c)$ is equivalent to the category $2Hilb$, first described by Baez in [3]. This category has 2-Hilbert spaces as objects, linear functors as 1-cells, and natural transformations as 2-cells. A 2-Hilbert space is an Abelian dagger category enriched over FHilb. For every 2-Hilbert space H, there is an isomorphism $H \cong FHilb^n$ for some n. Linear functors are functors that are completely determined by their value on basis elements. The structure of this proof was developed through discussions with Jamie Vicary.

By lemma 3.2.12, the hom-categories $2(FHilb)_c(A, B)$ are isomorphic to $FHilb^n$, where n is the dimension of $A \otimes B$. We can apply this lemma because $FHilb$ has biproducts, and the units of matrix algebras are copyable states. We will argue that this does not hold for arbitrary $0 - cells$.

A module of an arbitrary biproduct $A = \oplus A_i$ of matrix algebras defines a decomposition on the underlying Hilbert space. However, if A is not classical, this decomposition does not determine the module. For every summand A_i, which is a matrix algebra, the corresponding summand $\mathbf{M}_i$ of the module $\mathbf{M}$ is a collection of automorphisms, indexed by the elements of A_i. As $\mathbf{M}_i$ is linear in the elements of A_i, we can restrict to a collection of automorphisms, indexed by a basis of A_i. This can be any collection of morphisms that sums to the identity. This is exactly the case when $\mathbf{M}_i$ is a module on any classical structure of A_i. In $FHilb$, every n-dimensional biproduct of matrix algebras is isomorphic to a biproduct of n 1-dimensional matrix algebras. So the objects of $2(FHilb)(A, B)$ correspond to n-tuples of Hilbert spaces, where n is the dimension of $A \otimes B$. However, the morphisms between the objects do not correspond to n-tuples of morphisms, as these do not need to respect the decomposition of $A \otimes B$ into 1-dimensional subspaces.

Lemma 3.6.1. *Let A, B be 0-cells of $2(FHilb)_c$. The hom-category $2(FHilb)_c(A, B)$ is a 2-Hilbert space.*

Proof. We need to prove, that $2(FHilb)_c(A, B)$ is an Abelian category, enriched over $FHilb$. By lemma 3.2.12, $2(FHilb)_c(A, B) \cong FHilb^n$ for some n. This means that $2(FHilb)_c(A, B)$ is the category of tuples of Hilbert spaces $(H_1, ..., H_n)$, indexed by the copyable points of $A \otimes B$; and tuples of morphisms $(f_1, ..., f_n)$, where $H_i \xrightarrow{f_i} H_i'$. As a consequence, the hom-sets of $2(FHilb)_c(A, B)$ correspond to tuples of hom-sets of $FHilb$. We can define addition and scalar multiplication of morphisms entry wise. It follows that $2(FHilb)_c(A, B)$ is an Abelian category, enriched over $FHilb$ because $FHilb$ is an Abelian category, enriched over $FHilb$. □

Lemma 3.6.2. *Let $\mathbf{M} \in 2(FHilb)(A, B)$ be a 1-cell. The assignment $- \otimes_A \mathbf{M} : 2(FHilb)(I, A) \to 2(FHilb)(I, B)$ maps every 1-cell $\mathbf{K} \in 2(FHilb)(I, A)$, to the 1-cell $\mathbf{K} \otimes_A \mathbf{M} \in 2(FHilb)(I, B)$; and every 2-cell $\mathbf{K} \xrightarrow{g} \mathbf{K}' \in 2(FHilb)(I, A)$, to the morphism $g \otimes_A \mathrm{id}_M \in 2(FHilb)(I, B)$. This is a linear functor in $2Hilb$.*

Proof. We need to prove that $-\otimes_A \mathbf{M}$ preserves horizontal composition of 2-cells and identity 2-cells. Let $\mathbf{K} \xrightarrow{g} \mathbf{K}', \mathbf{K}' \xrightarrow{g'} \mathbf{K}'' \in 2(Fhilb)(I, A)$ be morphisms. We have the following equality, by the exchange law: $-\otimes_A \mathbf{M}(g' \circ g) = (g' \circ g) \otimes_A (\mathrm{id}_M) = (g' \otimes_A \mathrm{id}_M) \circ (g \otimes_A \mathrm{id}_M) = - \otimes_A \mathbf{M}(g') \circ - \otimes_A \mathbf{M}(g)$. The identity 2-cells are preserved by $- \otimes_A \mathbf{M}$ by lemma 3.2.6. This proves that $- \otimes_A \mathbf{M}$ is a functor. It is left to prove that it is a linear functor. In $Fhilb$, horizontal composition is given by generalised matrix multiplication. In this case $\mathbf{K} \otimes_A \mathbf{M}$ is the multiplication of the vector $\mathbf{K}$, with the matrix $\mathbf{M}$. This means that the image of any vector $\mathbf{K}$ under H is completely determined by the image of the entries of the vector; hence, H is a linear functor. □

Lemma 3.6.3. *Let $\mathbf{M} \xrightarrow{f} \mathbf{N} \in 2(Fhilb)(A, B)$ be a 2-cell. The collection of morphisms $\{\hat{f}_\mathbf{K} := \mathrm{id}_\mathbf{K} \otimes_A f\}$, indexed by the objects $\mathbf{K} \in 2(FHilb)(I, A)$, is a natural transformation from $- \otimes_A \mathbf{M}$ to $- \otimes_A \mathbf{N}$.*

Proof. We know that f and $\mathrm{id}_{\mathbf{K}}$ are morphisms in $2(Fhilb)(A,B)$, and $2(Fhilb)(I,A)$, respectively, for all $\mathbf{K} \in 2(Fhilb)(I,A)$. By the definition of horizontal composition 3.2.5, $\hat{f}$ is a collection of morphisms in $2(FHilb)(I,B)$, from $(- \otimes_A \mathbf{M})(2(FHilb)(I,A))$ to $(- \otimes_A \mathbf{N})(2(FHilb)(I,A))$. It is left to show, that $\hat{f}_{\mathbf{K}}$ is natural in $\mathbf{K}$. In other words, for every morphism $\mathbf{K} \xrightarrow{g} \mathbf{K}'$, the diagram below commutes:

$$\begin{array}{ccc} \mathbf{K} \otimes_A \mathbf{M} & \xrightarrow{\hat{f}_{\mathbf{K}}} & \mathbf{K} \otimes_A \mathbf{N} \\ {\scriptstyle (-\otimes \mathbf{M})(g)} \downarrow & & \downarrow {\scriptstyle (-\otimes \mathbf{N})(g)} \\ \mathbf{K}' \otimes_A \mathbf{M} & \xrightarrow[\hat{f}_{\mathbf{K}'}]{} & \mathbf{K}' \otimes_A \mathbf{N} \end{array} \tag{3.71}$$

Unfolding definitions, $(- \otimes \mathbf{M})(g) := g \otimes_A \mathrm{id}_M$, $(- \otimes \mathbf{N})(g) := g \otimes_A \mathrm{id}_N$, $\hat{f}_{\mathbf{K}} := \mathrm{id}_K \otimes_A f$, and $\hat{f}_{\mathbf{K}'} := \mathrm{id}_{K'} \otimes_A f$. The diagram commutes because $(g \otimes_A \mathrm{id}_{\mathbf{N}}) \circ (\mathrm{id}_{\mathbf{K}} \otimes_A f) = (\mathrm{id}_{\mathbf{K}'} \otimes_A f) \circ (g \otimes_A \mathrm{id}_{\mathbf{M}})$. □

Lemma 3.6.4. *There exists an isomorphism $H : 2(FHilb_c) \to 2Hilb$. The functor H maps every 0-cell A, to the hom-category $2(Hilb)_c(I,A)$. It maps every 1-cell $\mathbf{M} \in 2(FHilb)_c(A,B)$ to the linear functor $- \otimes_A \mathbf{M}$ from $2(Hilb)_c(I,A)$ to $2(Hilb)_c(I,B)$, and every 2-cell $\mathbf{M} \xrightarrow{f} \mathbf{N}$ to the natural transformation $\tilde{f}$ with components $\tilde{f}_{\mathbf{K}} = \mathrm{id}_{\mathbf{K}} \otimes_A f$ between the linear functors $H(\mathbf{M})$ and $H(\mathbf{N})$.*

Proof. First, we will prove that H is a functor from $2(FHilb)$ to $2Hilb$, by showing that is preserves horizontal composition of 1- and 2-cells, vertical composition of 2-cells and identities of both 1- and 2-cells. After this, we will prove that the functor H is well-defined on 0-,1- and 2-cells. Then, we will prove that H is surjective up to isomorphism on 0-cells, and full on hom-sets of 0- and 1-cells of $2(FHilb)$. Finally, we will prove that H is injective up to isomorphism on 0-cells, and and faithful on hom-sets of 0- and 1-cells.

We will show that H preserves horizontal composition of 1-cells and 2-cells. Let $\mathbf{M} \in 2(Fhilb)(A,B)$, and $\mathbf{N} \in 2(FHilb)(B,C)$ be 1-cells. By definition we have $H(\mathbf{M} \otimes_B \mathbf{N})(-) = - \otimes_A (\mathbf{M} \otimes_B \mathbf{N})$. By the 2-cell $\alpha^{-1}_{-,M,N}$, this is isomorphic to $(- \otimes_A \mathbf{M}) \otimes_B \mathbf{N}$, which equals $(H(\mathbf{M}) \otimes_B H(\mathbf{N}))(-)$.

Let $\mathbf{M} \xrightarrow{h} \mathbf{M}' \in 2(FHilb)(A,B)$, and $\mathbf{N} \xrightarrow{k} \mathbf{N}' \in 2(FHilb)(B,C)$ be 2-cells. $H(h \otimes_B k) = (h \hat{\otimes}_B k)$, which is the collection of morphisms $(h \hat{\otimes}_B k)_{\mathbf{K}} = \mathrm{id}_{\mathbf{K}} \otimes_A (h \otimes_B k)$. This is again by $\alpha_{-,M,N}$ isomorphic to $(\mathrm{id}_{\mathbf{K}} \otimes_A h) \otimes_B k$. This is $\bar{h}_{\mathbf{K}} \otimes_B \bar{k}_{\mathbf{K}\otimes\mathbf{M}}$, so $H(h \otimes_B k) = H(h) \otimes_B H(k)$.

Furthermore, H preserves vertical composition of 2-cells, as for the 2-cells $\mathbf{M} \xrightarrow{h} \mathbf{M}' \xrightarrow{h'} \mathbf{M}''$ of $2(FHilb)$, $H(h' \circ h) = (h' \hat{\circ} h)$. This is the natural transformation with components $\mathrm{id}_{\mathbf{K}} \otimes_A (h' \circ h)$. By the exchange law, this equals $(\mathrm{id}_{\mathbf{K}} \otimes_A h') \circ (\mathrm{id}_{\mathbf{K}} \otimes_A h)$ for all 1-cells $\mathbf{K} \in 2(FHilb)(I, A)$. It follows that $H(h' \circ h) = H(h') \circ H(h)$.

The functor H is well-defined on 0-cells by lemma 3.6.1; It is well-defined on 1-cells by lemma 3.6.2; and it is well-defined on 2-cells by lemma 3.6.3.

Moreover, every 0-cell of $2Hilb$ is isomorphic to $Fhilb^n$, for some n. By lemma 3.2.12, this is isomorphic to $H(A)$, where A is a classical structure on an n-dimensional Hilbert space. This means that H is surjective, up to isomorphisms, on 0-cells.

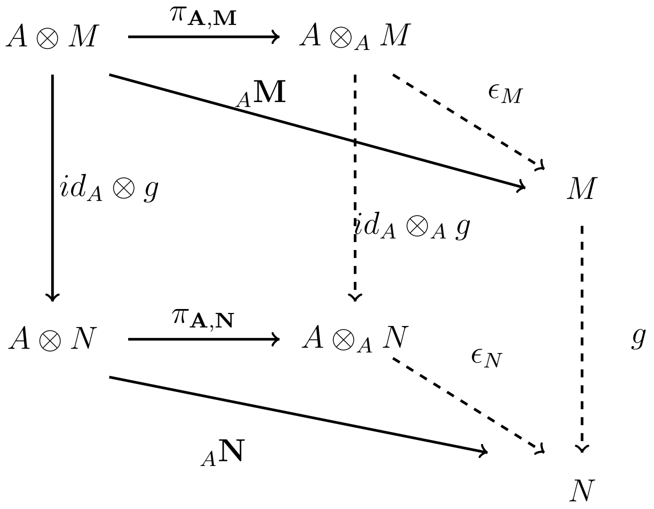
We will prove now, that H is full on hom-sets of 0-cells, up to isomorphism. Let A, B be 0-cells of $2Hilb$. We know that $A \cong FHilb^n$ and $B \cong FHilb^m$, for some n, m. A linear functor f from A to B is a collection of functors $f_i : FHilb \xrightarrow{F} Hilb^m$ for $i = 1, ..., n$, so it is a matrix of functors $f_{i,j} : FHilb \xrightarrow{F} Hilb$. Every functor $f_{i,j}$ is completely determined by its value on $\mathbb{C}$. It follows that $f_{i,j}(K) = K \otimes f_{i,i}(\mathbb{C})$ for any Hilbert space $K \in FHilb$, so $f_{i,j} = - \otimes f_{i,j}(\mathbb{C})$. Note that the matrix of Hilbert spaces $(f_{i,j}(\mathbf{C}))_{(i,j)}$ defines a module on $A \otimes B$, by 3.2.12. It follows that f is the image of the matrix of Hilbert spaces $(f_{i,j}(\mathbf{C}))_{(i,j)}$, under H. As a result, H is surjective up to isomorphisms on 1-cells.

Now we will prove that H is full on hom-sets of 1-cells. Let n be a natural transformation in $2Hilb$ between linear functors $H(\mathbf{M})$ and $H(\mathbf{N})$, for $\mathbf{M}, \mathbf{N} \in 2(FHilb)(A, B)$. In that case, n is a collection of morphisms in $2(FHilb)(I, B)$, from $(- \otimes_A \mathbf{M})(2(FHilb)(I, A))$ to $(- \otimes_A \mathbf{N})(2(FHilb)(I, A))$, indexed by a basis of $2(FHilb)(I, A)$. So it is a collection of linear maps $n_{\mathbf{K}} : \mathbf{K} \otimes_A \mathbf{M} \to \mathbf{K} \otimes_A \mathbf{N}$ for every $\mathbf{K} \in 2(FHilb)(I, A)$. Every $n_{\mathbf{K}}$ is a sum of linear maps $n_{\mathbf{K_i}}$, indexed by the basis of $2(FHilb)(I, A)$ as $\mathbf{K}$ is a biproduct of n Hilbert spaces. Every linear map $n_{\mathbf{K_i}}$ is a collection of linear maps $n_{\mathbf{K_{i,j}}}$ indexed by a basis of $2(FHilb)(I, B)$ as $\mathbf{K} \otimes_A \mathbf{M}$ is a biproduct of m Hilbert spaces. So $n_{\mathbf{K}}$ is a matrix $n_{\mathbf{K_{i,j}}}$ of Hilbert spaces indexed by the bases of $2(FHilb)(I, A)$ and $2(FHilb)(I, B)$.

As n is natural in $\mathbf{K}$, for any morphism $\mathbf{K} \xrightarrow{g} \mathbf{K}'$ in $2(FHilb)(I, A)$, we have the equality $g \otimes_A \mathrm{id}_{\mathbf{N}_{j,k}} \circ n_{\mathbf{K}_{i,j}} = n_{\mathbf{K}'_{i,j}} \circ (g \otimes_A \mathrm{id}_{\mathbf{M}_{j,k}})$. This is true, in particular when $\mathbf{K}'_{i,j} = \mathbb{C}$. We know that $n_{\mathbb{C}_{i,j}} = (\mathrm{id}_{\mathbb{C}} \otimes n_{\mathbb{C}_{i,j}})$. We use this to see that the equality $n_{\mathbb{C}_{i,j}} \circ (g \otimes \mathrm{id}_{M_{j,k}}) = (g \otimes \mathrm{id}_{N_{j,k}}) \circ n_{\mathbf{K}_{i,j}}$, is equivalent to $g \otimes n_{\mathbb{C}_{i,j}} = (g \otimes \mathrm{id}_{N_{j,k}}) \circ n_{\mathbf{K}_{i,j}}$. It follows that $n_{\mathbf{K}_{i,j}} = \mathrm{id}_{\mathbf{K}_{i,j}} \otimes n_{\mathbb{C}_{i,j}}$ because $g \otimes \mathrm{id}_{\mathbf{N}_{j,k}}$ is surjective. It follows that $H(n_{\mathbb{C}}) = n$, where $n_{\mathbb{C}} = \sum_{i,j} n_{\mathbb{C}_{i,j}}$, so H is full on hom-sets of 1-cells.

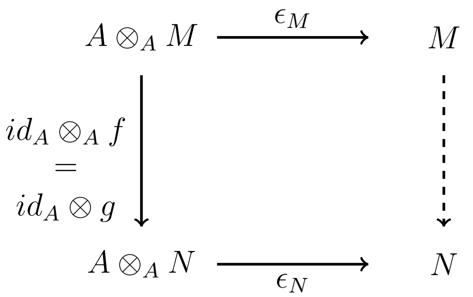
At last we will prove that H is faithful on hom-sets of 1-cells and 0-cells, and that H is injective on 0-cells.

The functor H is faithful on hom-sets of 1-cells because $H(f) = H(g)$, for two 2-cells $f, g \in 2(FHilb)(\mathbf{M}, \mathbf{N})$, implies that $\mathrm{id}_{\mathbf{K}} \otimes_A f = \mathrm{id}_{\mathbf{K}} \otimes_A g$, for all $\mathbf{K} \in 2(FHilb)(I, A)$. In particular, this holds for the module $\mathbf{A}$, defined in lemma 3.2.7. By this lemma we have the following commuting three-dimensional diagram:



(3.72)

The square at the back commutes by definition 3.2.5, the upper and lower triangle commute by lemma 3.2.7, the square in the front commutes because $\mathbf{M} \xrightarrow{g} \mathbf{N}$ is a module homomorphism. It follows that the square on the right commutes as well. We have a similar diagram for f. This implies that both f and g make the diagram below commute:



(3.73)

As this is a unique morphism, we know that $f = g$, so H is faithful on hom-sets of 1-cells.

Now we will prove that H is faithful on hom-sets of 0-cells. Let $\mathbf{M}, \mathbf{N} \in 2(FHilb)(A, B)$ be 1-cells. Suppose that $H(\mathbf{M}) \cong H(\mathbf{N})$. That means that there exists a natural isomorphism n, between $H(\mathbf{M})$ and $H(\mathbf{N})$ in $2FHilb$. By surjectivity on 2-cells of H, there exists an isomorphism between the modules $\mathbf{M}$ and $\mathbf{N}$ in $2(FHilb)$. So H is injective on 1-cells.

Now we will prove that H is injective on 0-cells. Suppose that $2(FHilb)(A, I) \cong 2(FHilb)(B, I)$, then there exists a linear functor $2(FHilb)(I, A) \xrightarrow{M} 2(FHilb)(I, B)$, and a linear functor $2(FHilb)(I, B) \xrightarrow{M^{-1}} 2(FHilb)(I, A)$, such that $M^{-1} \circ M = \mathrm{id}_{2(FHilb)(I,A)}$, and $M \circ M^{-1} = \mathrm{id}_{2(FHilb)(I,B)}$. Because H is surjective on 1-cells, there exist 1-cells $\mathbf{M} \in 2(FHilb)(A, B)$ and $\mathbf{M^{-1}} \in 2(FHilb)(B, A)$, such that $M = H(\mathbf{M})$, and $M^{-1} = H(\mathbf{M^{-1}})$.

Now we will prove that $\mathbf{M} \otimes_{\mathbf{B}} \mathbf{M}^{-1} = \mathbf{B}$. We know that $H(\mathbf{M} \otimes_{\mathbf{B}} \mathbf{M}^{-1}) = \mathrm{id}_{2(FHilb)(I,B)}$ because H preserves horizontal composition of 1-cells. As a consequence, $\mathbf{M} \circ \mathbf{M}^{-1} = \mathbf{B}$, as H preserves identity 1-cells, and H is faithful on 1-cells. Similarly we can prove that $\mathbf{M}^{-1} \otimes_{\mathbf{A}} \mathbf{M} = \mathbf{A}$, so $A \cong B$. □

By lemma 3.4.6, $2(FHilb)$ is compact. The adjoint defines a dagger functor on every Hilbert space. The dual of a module is given by the conjugation of the corresponding matrix of Hilbert spaces. By 3.4.7, all hom-sets have biproducts, these are the entry wise direct sums of Hilbert spaces.

3.7 2(**Rel**)

In chapter 2, we discussed Frobenius algebras in Rel, which are groupoids. All Frobenius algebras are normal and commutative Frobenius algebras correspond to Abelian groups. In this section, we will explore the structure of modules on Frobenius algebras in *Rel* and we will discuss the category $2(Rel)$.

3.7.1 Bimodules of groupoids

Recall that in *Rel*, dagger biproducts are disjoint unions of sets. Frobenius algebras in *Rel* are groupoids. These are categories of which every morphism has an inverse. *Rel* is compact, hence the tensor distributes over the biproduct. As a result, we can regard any groupoid as a biproduct of indiscrete groupoids. The unit u_A of a groupoid A is given by the disjoint union of units of each of the indiscrete groupoids. Recall that by section 2.4.4, this is the relation $\{\mathrm{id}_* \sim \mathrm{id}_a | a \in \mathrm{Ob}(A)\}$, between the identity morphism of I and the identity morphisms of a groupoid A. We will write $a, b, c, ...$ for the objects of a groupoid A; $\mathrm{id}_a, \mathrm{id}_b, \mathrm{id}_c, ...$ for the corresponding identity morphisms on these objects; and $\gamma, \delta, ...$ for the morphisms.

We will prove that a module $\mathbf{M}$ of a groupoid A on the set M defines a groupoid on M. The collection of objects of this induced groupoid is $\{M_{\mathrm{id}_a} | a \in \mathrm{Ob}(A)\}$; the morphisms are given by $M_{\mathrm{id}_a} \xrightarrow{\mathbf{M}_\gamma} M_{\mathrm{id}_b}$ for $a \xrightarrow{\gamma} b \in A$.

$\mathbf{M}_{\mathrm{id}_a}$ is the identity morphism on M_{id_a}, since id_a is an identity morphism and by 3.4. $\mathbf{M}_\gamma$ defines a bijection between M_{id_a} and M_{id_b} with inverse $\mathbf{M}_{\gamma^{-1}}$ since $\mathbf{M}_\gamma \circ \mathbf{M}_{\gamma^{-1}} = \mathbf{M}_{\gamma \circ \gamma^{-1}} = \mathbf{M}_{\mathrm{id}_a}$ by 3.4. $\mathbf{M}_\gamma$ is empty on $M \setminus M_{\mathrm{id}_a}$ because $\mathbf{M}_\gamma(\mathbf{M}_{\mathrm{id}_c}) = \mathbf{M}_{\gamma \circ \mathrm{id}_c} = \emptyset$ because $a \circ \mathrm{id}_c$ is not defined for any identity morphism $\mathrm{id}_c \neq \mathrm{id}_a$. Note that this means that $\mathbf{M}_\gamma$ is an injective function for all $\gamma \in A$.

If we write $\#S$ for the number of elements of the set S, $\#M_{\mathrm{id}_a} = \#M_{\mathrm{id}_{a'}}$ for all identity morphisms $\mathrm{id}_a, \mathrm{id}_{a'}$ of an indiscrete groupoid. So if $\mathbf{M}$ is a module of an indiscrete groupoids A on M, the number of elements of M is divisible by the number of objects of A.

Module homomorphisms $\mathbf{M} \xrightarrow{f} \mathbf{M}'$ in *Rel* correspond to groupoid homomorphisms. To prove this, we need to show that f respects the groupoid structure, if and only if it is a module homomorphism. This is straightforward, since for every morphism $\gamma \in A$, $f \circ \mathbf{M}_\gamma = f \circ \mathbf{M} \circ (\gamma \otimes \mathrm{id}_M) = \mathbf{M}' \circ (\gamma \otimes f) = \mathbf{M}'_\gamma \circ (\mathrm{id}_A \otimes f)$. The second equality holds if and only if f is a module homomorphism. The other equalities always hold. So $f(\mathbf{M}_\gamma) = \mathbf{M}'_\gamma$ if and only if f is a module homomorphism.

This corresponds to our earlier result in section 3.2.4, that a module $\mathbf{M}$ of a dagger biproduct of Frobenius algebras $A = \oplus_i A_i$ with unit $u_A = \sum_i u_{A_i}$, is the biproduct $\mathbf{M} = \oplus_i \mathbf{M}_{u_{A_i}}$. The underlying space M is the biproduct $\oplus_i M_{u_{A_i}}$. Furthermore, a module homomorphism f is the sum of module homomorphisms $\mathbf{M}_{u_{A_i}} \xrightarrow{f_i} \mathbf{M}'_{u_{A_i}}$.

Note that $M_{u_{A_i}}$ can be empty, as $\mathbf{M}$ is a relation.

We can classify all modules $\mathbf{M}$ of a groupoid A. For every indiscrete sub-groupoid $A_i \subset A$, we have a surjective groupoid homomorphism $\phi_i : A_i \to M_{u_{A_i}}$, where $\phi_i(\gamma) = \mathbf{M}_\gamma$. This is a groupoid homomorphism, since $\phi_i(\gamma \circ \delta) = \mathbf{M}_{\gamma\circ\delta} = \mathbf{M}_\gamma \circ \mathbf{M}_\delta = \phi_i(\gamma) \circ \phi_i(\delta)$. It is surjective, since $\mathbf{M}_\gamma(M)$ is defined for all $\gamma \in A_i$. It follows that the induced groupoid on $M_{u_{A_i}}$ is isomorphic to the quotient groupoid $A_i/\sim$ where $\gamma \sim \delta$ if $\phi(\gamma) = \phi(\delta)$.

To summarise, we can describe each module $\mathbf{M}$ of A on M as follows: If $A = \oplus_{i\in I} A_i$, then $\mathbf{M}$ defines the groupoid $\oplus_{j\in J} \mathbf{M}_j$ on M, for some $J \subset I$. Each $\mathbf{M}_j$ is isomorphic to some quotient groupoid of A_j. This implies that the hom-sets $2(Rel)(A, B)$ are isomorphic to the collection of all sets with the structure of sub groupoids of $A \otimes B$. This collection is a category, where the morphisms are the relations that respect the groupoid structure.

3.7.2 $2(Rel)$

By section 3.2.4, horizontal composition is given by abstract matrix multiplication. Let $\mathbf{M} \in 2(Rel)(A, B)$ and $\mathbf{N} \in 2(Rel)(B, C)$ be 1-cells and let $A = \oplus_i A_i$, $B = \oplus_j B_j$, $C = \oplus_k C_k$ be 0-cells, which are direct sums of indiscrete groupoids. Let $\{a_i\}, \{b_j\}, \{c_k\}$ be the units of A_i, B_j, and C_k respectively. The horizontal composition $\mathbf{M} \otimes_B \mathbf{N}$ of the modules $\mathbf{M}, \mathbf{N}$ can be expressed in terms of their induces groupoids as $(\mathbf{M} \otimes_B \mathbf{N})_{i,k} = \sum_j \mathbf{M}_{a_i,b_j} \otimes \mathbf{N}_{b_j,c_k}$.

By the previous subsection, the hom-sets $2(Rel)(A, B)$ are isomorphic to the category of all sets with the additional structure of a sub-groupoid of $A \otimes B$ and relations between these sets that have the structure of groupoid homomorphisms.

When A and B are classical structures, the only subgroupoid of $A \otimes B$ is $A \otimes B$ itself, which is a disjoint union of 1-element groups. It follows that $2(Rel)_c(A, B) \cong Rel^n$ for all classical structures A, B, and for $n = \#(A \otimes B)$. Note that this corresponds to lemma 3.4.7.

There is a one-to-one correspondence between the 0-cells $A \in 2(Rel)_c$ and the hom-categories $2(Rel)_c(I, A) \cong Rel^n$, where n is the number of objects of A. Every module $\mathbf{M} \in 2(Rel)(A, B)$ corresponds to a functor $- \otimes_A \mathbf{M} : 2(Rel)_c(I, A) \to 2(Rel)_c(I, B)$ and every module homomorphism $\mathbf{M} \xrightarrow{f} \mathbf{N}$ corresponds to a natural transformation between the functors $- \otimes_A \mathbf{M}$ and $- \otimes_A \mathbf{N}$. This correspondence provides us with a definition of the category $2Rel$, which is analogous to $2Hilb$. This is the category that has tuples of sets as 0-cells, matrices of sets as 1-cells, and matrices of relations as 2-cells. We will omit the details because this is very similar to lemma 3.6.4.

Rel has a dagger functor, which is the identity, so the same is true for $2(Rel)$. *Rel* has biproducts given by disjoint unions of sets, hence the hom-sets $2(Rel)(A, B)$ also have biproducts, given by disjoint unions of bimodules of A and B. *Rel* is compact and all Frobenius algebras in *Rel* are special; therefore, $2(Rel)$ is also compact.

Chapter 4

Conclusion

In this dissertation, we have discussed several categorical models of quantum theory. In this section, we will give a brief summary and comparison of these different models. Furthermore, we will discuss the results we derived about 2-categorical models for quantum theory, and how this contributes to the current research in this field. Finally, we will point out the open problems that are still present, and we will give some suggestions for further research.

4.1 A comparison of categorical models for quantum theory

4.1.1 Monoidal categories

The initial categorical model for quantum theory was given by monoidal dagger compact categories. This is an expressive model for pure state quantum theory, in which most features of Hilbert spaces arise naturally. Variations on monoidal dagger compact categories have been proposed, in order to express mixed states and classical states explicitly. The most interesting ones are $CPM(\mathbf{C})$, $CP^{\oplus}(\mathbf{C})$, $Split^{\dagger}(CPM(\mathbf{C})$, and $CP^{*}(\mathbf{C})$.

The category $CPM(\mathbf{C})$ is the category of mixed states and completely positive morphisms of a monoidal category $\mathbf{C}$. This is a model for mixed state quantum theory. It is a generalisation of density matrices and completely positive operators on Hilbert spaces.

The category $CP^{\oplus}(\mathbf{C})$ is the biproduct completion of $CPM(\mathbf{C})$. It exploits the fact that in $FHilb$, classical information corresponds to biproducts of the unit object. As we have seen, it allows us to describe classical and quantum information explicitly. Moreover, we can reason about quantum processes as a whole, including the branching due to measurements.

The category $Split^{\dagger}(CPM(\mathbf{C}))$, obtained by freely splitting idempotents of $CPM(\mathbf{C})$, is based on the correspondence between classical or quantum information, and measurements. This category also describes classical and quantum systems explicitly. It is more elegant than $CP^{\oplus}(\mathbf{C})$ because $CP^{\oplus}(\mathbf{C})$ requires the additional structure of biproducts, while all the structure required for $Split^{\dagger}(CPM(\mathbf{C}))$ is present in $\mathbf{C}$.

The category $CP^{*}(\mathbf{C})$ is the category of dagger Frobenius algebras of $\mathbf{C}$. This is based on the observation that in *FHilb* all quantum systems correspond to dagger Frobenius algebras, and that the ones that correspond to classical systems are commutative. We can clearly distinguish classical from quantum information, yet both are described by the same structure. This category implicitly captures a generalisation of the no-broadcasting theorem because the multiplication of commutative dagger Frobenius algebras is a morphism of $CPM(\mathbf{C})$, while the multiplication of a non-commutative dagger Frobenius algebras is not.

It was shown that $Split^{\dagger}(CPM(\mathbf{C}))$ and $CP^{*}(\mathbf{C})$ have a richer structure than $CP^{\oplus}(\mathbf{C})$, when $\mathbf{C}$ is the category *Rel*. On the other hand, when $\mathbf{C} = FHilb$, all three models are equivalent. In other words, all three models are all sufficient to describe quantum processes. Nevertheless, the first two categories are more interesting. It is still an open question whether or not $CP^{*}(\mathbf{C})$ is equivalent to $Split^{\dagger}(CPM(\mathbf{C}))$. If this is not the case, that would imply that there exist different types of systems, in addition to classical and quantum systems.

The interpretation of the categories $CPM(\mathbf{C}), CP^{\oplus}(\mathbf{C})$, $Split^{\dagger}(CPM(\mathbf{C}))$, and $CP^{*}(\mathbf{C})$ are a bit problematic. This is due to the fact that in all these categories, physical systems are described as operators. Mathematically this works very well, yet it is not very intuitive. However, this point of view is generally accepted amongst physicists.

Ordinary monoidal dagger compact categories have the advantage that the interpretation is very clear. Objects correspond to physical objects, morphism correspond to operations, tensor products represent spatial separation and composition of morphisms correspond to the subsequent execution of operations. Furthermore, it is a simple category that requires little structure, but it is very expressive. In a way, this captures all variations on monoidal categories implicitly.

4.1.2 Weak 2-categories

An alternative categorical model for quantum theory is given by weak 2-categories. This model is different from the monoidal categorical models with respect to its interpretation; in its structure, however, it is a generalisation of these models.

Ordinary monoidal dagger compact categories correspond to interpretation of quantum theory, which explains measurements as a collapse of wave functions. the 2-categorical models correspond to the many worlds and decoherence interpretation. The embedding of monoidal categories in 2-categories demonstrates how these different interpretations are related.

The interpretation of $2(\mathbf{C})$ is intuitive according to the many worlds interpretation. 1-cells correspond to physical objects, which interact with one or more separate systems. We can regard such a separate system as the environment of the object. By this interaction, information about the state of the physical object is transferred to the environment. If the object is in a superposition, this may result in a superposition of the joint system of the object and its environment. Due to this transfer of information, the physical object may behave classical relative to its environment. It is less clear how a physical system can interact with two separate environments at the same time. Horizontal composition of 1-cells corresponds to a spatial separation of two physical objects, which interact with the same environment. Morphisms are interpreted as operations on physical objects, and vertical composition of morphisms corresponds to a subsequent execution of operations.

We have shown that the hom-category of scalars of the 2-category $2(\mathbf{C})$ is isomorphic to $\mathbf{C}$ for any monoidal category $\mathbf{C}$. This means that everything that can be expressed in $\mathbf{C}$, can also be expressed in $2(\mathbf{C})$. The construction $2(-)$ provides a generalisation for any symmetric monoidal dagger category, where all coequalisers exist, including the ones we have discussed in chapter 2.

The motivation for this framework is similar to that of $Split^{\dagger}(CPM(\mathbf{C}))$. Both models use the correspondence between classical/quantum information and measurements. As we have proven, measurement outcomes in $2(\mathbf{C})$ have a canonical dagger Frobenius algebra. We do not know if this is the case in $Split^{\dagger}(CPM(\mathbf{C})$. Also, there are similarities between $2(\mathbf{C})$ and $CP^{\oplus}(CP(\mathbf{C}))$. Both categories can be used to describe quantum protocols as a whole, independent of branching due to measurements. However, the structure of $2(\mathbf{C})$ is much richer. This is based on Frobenius algebras and modules; these correspond to biproducts in Hilbert spaces, but in other categories, such as *Rel*, these correspond to richer structures.

We conclude that, depending on the practical application, some models might be better than others. Overall, the original monoidal dagger compact category is the most simple model with the most intuitive interpretation, while being just as expressive as the other categories. The categories $CP^*(\mathbf{C})$ and $2(\mathbf{C})$ are the most suitable categories to describe the interaction between classical and quantum information. The category $2(\mathbf{C})$ has the advantage, that it can describe entire protocols, independent of branching due to measurement. Furthermore, it has a richer diagrammatic language. As we have discussed, $CP^*(\mathbf{C})$ gives a clear distinction between classical and quantum systems. However, one may argue that this category is bigger than necessary, as it contains an object for every dagger Frobenius algebra. In $2(\mathbf{C})$, dagger Frobenius algebras are operations on 1-cells, not 1-cells by themselves. However, different dagger Frobenius algebras in $2(\mathbf{C})(I, I)$ correspond to different, but isomorphic

1-cells. These are different horizontal compositions of 1-cells, and can be distinguished in the diagrammatic language.

4.2 Our contribution

In this dissertation, we have proposed a module-theoretic construction $2(-)$, which turns monoidal categories into 2-categories. We have shown that the hom-category of scalars of our model is isomorphic to the original monoidal dagger category $\mathbf{C}$. Furthermore, this construction preserves dagger functors, compactness, and biproducts. In addition, we described which operations in this category correspond to measurements, and we proved that these operations are unitary. Also, we have proposed a general definition for dagger Frobenius algebras in weak 2-categories. We have characterised a class of 1-cells that have canonical dagger Frobenius algebras. Based on this class, we have constructed a new 2-category $Sym(2\mathbf{C})$, in which all 1-cells can be deleted. This 2-category provides the mathematical foundation for the semantical description of weak 2-categories given in [20]. This model provides us with a new perspective on quantum theory; and connects categorical quantum theory in terms of monoidal categories to topological quantum field theory and representation theory, which is also described in [20].

To illustrate our model, we have worked out two interesting and important examples. As we have shown in section 3.6, when we apply this construction the category $FHilb$, we obtain a category that is richer in structure than $2Hilb$. The restriction to $2(FHilb)_c$ is isomorphic to $2Hilb$. Finally, we have given a classification of dagger Frobenius algebras and modules of dagger Frobenius algebras in Rel, the model for classical possibilistic computation. We have used this to give a description of the category $2(Rel)$, and a definition of $2Rel$, analogous to $2Hilb$.

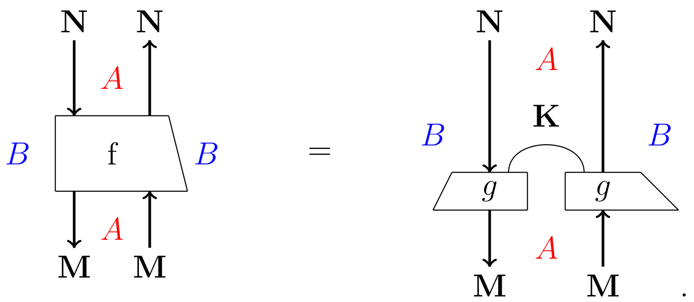
4.3 Further research

There are several remaining open problems and interesting questions for further research. We will discuss some of them, regarding the standard model for quantum theory, $FHilb$, the relation between $2(\mathbf{C})$ and other categorical models, the mathematical properties of the abstract category $2(\mathbf{C})$, and possible applications of this model.

While we have given a detailed description of $2(FHilb)_c$, It is not very clear how we should interpret the hom-categories $2(FHilb)(A, B)$, if A, B are not classical structures. It would be interesting to investigate the relation between $2(FHilb)$ and $2Hilb$ further; whether this is an equivalence, or if we can find a high-level explanation why this cannot be the case.

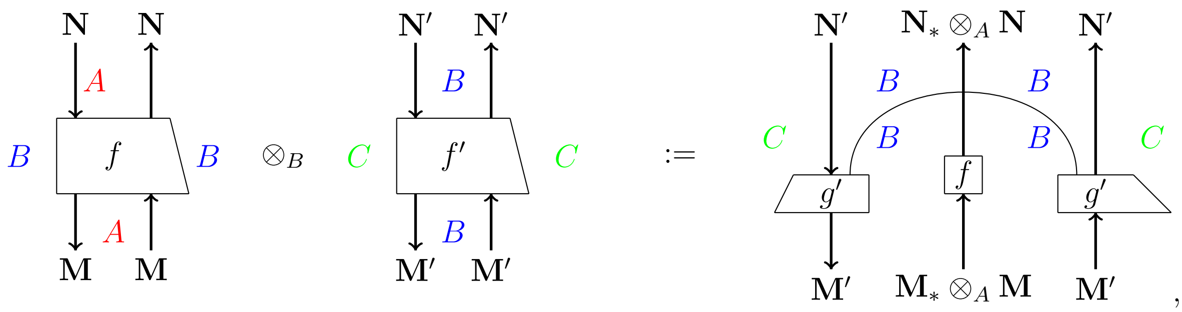
The relation between $2(\mathbf{C})$, and $CP^*(\mathbf{C})$ or $Split^\dagger(CP(\mathbf{C}))$ could be explored further. We have tried to formulate a generalisation of $CP(-)$ and $CP^*(-)$ on 2-categories. So far, this did not have any interesting result. The problem is that horizontal composition with respect to any 0-cell other than I is not commutative. A counter example is horizontal composition in $FHilb$. This corresponds to matrix multiplication, which is non-commutative. As a result, there is no obvious way to define horizontal composition of dagger Frobenius algebras in $2(\mathbf{C})$. Similarly, the 2-categorical analogue of completely positive maps cannot be composed horizontally, as this necessarily involves a symmetry morphism. This is illustrated in the pictures below.

Analogous to completely positive morphisms in $\mathbf{C}$, a 2-cell f is a completely positive, if and only if there exists a 2-cell $g \in 2(\mathbf{C})(A, B)$ and a 1-cell $\mathbf{K} \in 2(\mathbf{C})(A, A)$, such that



(4.1)

Composition of completely positive 2-cells f and f' would be



(4.2)

which is not defined.

So far, we have focused on the mathematical foundation of weak 2-categories. Now that we have established basic results about our mathematical framework, it would be interesting to see how we can formulate quantum protocols in terms of our 2-category $2(\mathbf{C})$. This has been done in in the specific case of $2Hilb$, and in the diagrammatic language of weak 2-categories in [20]. It would be interesting to see how the topological results about quantum protocols described in [20] translate to the module-theoretic context. The framework that we have developed makes it possible to explore these quantum protocols in other categories, such as $2(Rel)$.

Furthermore, we would like to know whether or not we can prove that the category $2(\mathbf{C})$ is a symmetric monoidal weak 2-category. Lastly, we would like to determine which class or weak 2-categories our formalism captures. In other words: Under which conditions is a weak 2-category $\mathbf{D}$ isomorphic to the weak 2-category $2(\mathbf{D})(I, I)$?

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